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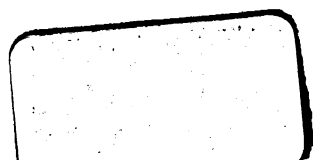
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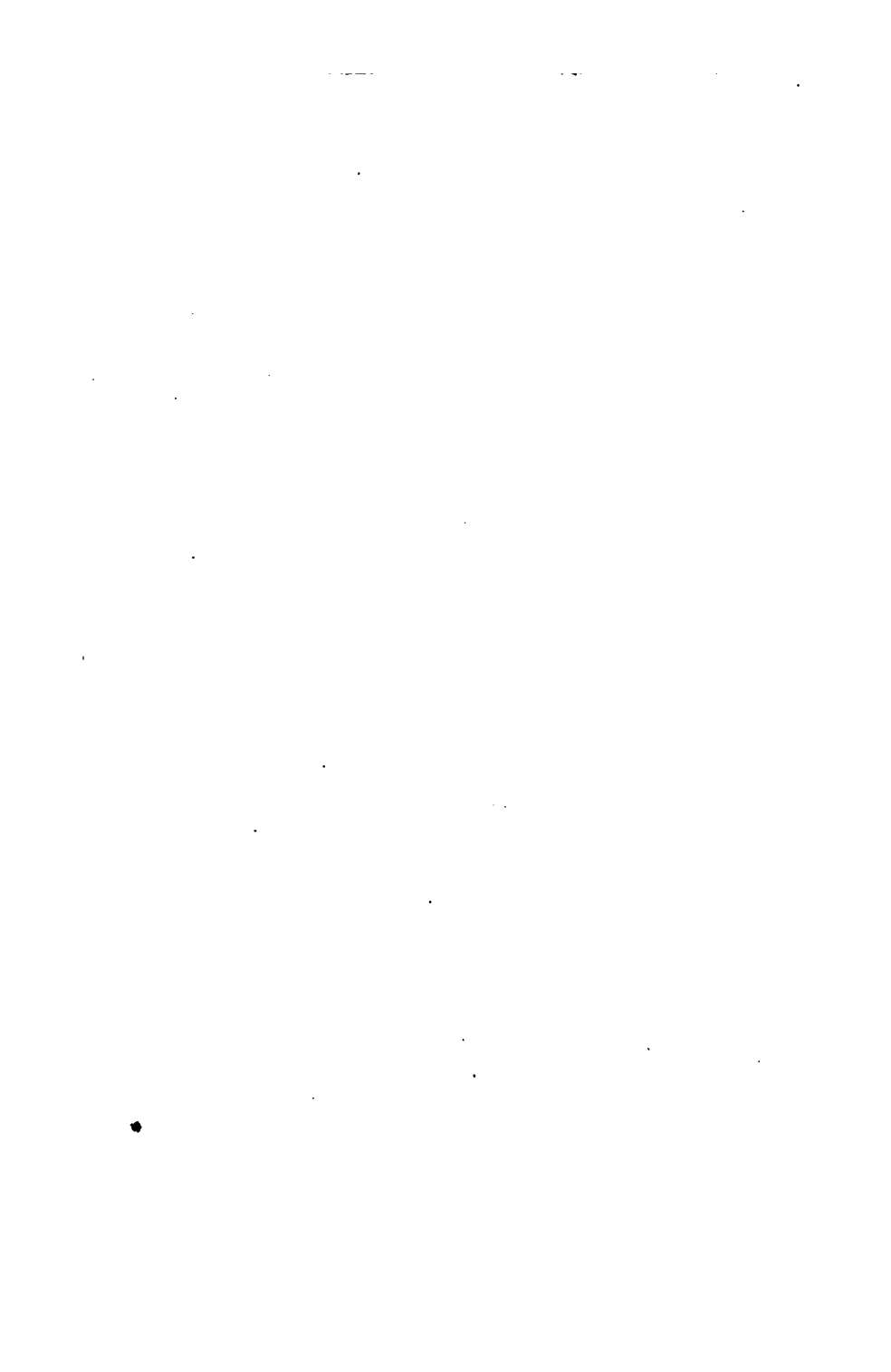
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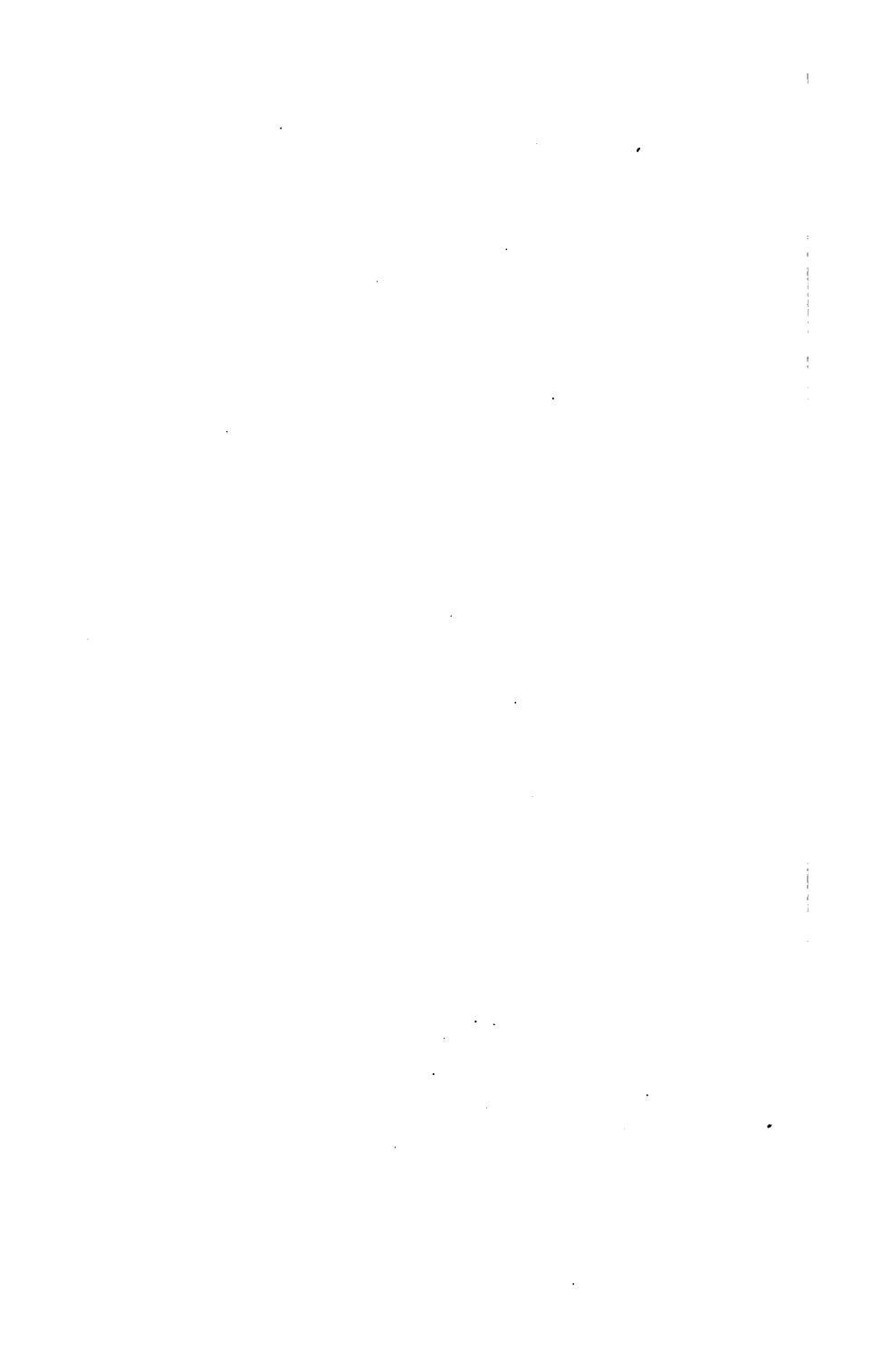
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# E U C L I D ' S E L E M E N T S   O F   G E O M E T R Y ,

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D E S I G N E D   F O R   T H E   U S E   O F   T H E   J U N I O R   C L A S S E S   I N   P U B L I C   A N D  
P R I V A T E   S C H O O L S .

B Y

R O B E R T   P O T T S ,   M . A .

T R I N I T Y   C O L L E G E .

*C O R R E C T E D   A N D   I M P R O V E D .*

L O N D O N :

L O N G M A N ,   G R E E N ,   L O N G M A N ,   R O B E R T S ,   &   G R E E N .

—  
M D C C C L X I V .

~~180. c. 172.~~  
183. a. 4.

**INTERNATIONAL EXHIBITION,**

1862.

A Medal has been awarded to R. Potts,  
"For the excellence of his Works on Geometry."

*Jury Awards, Class XXIX, p. 313.*



## PREFACE TO THE THIRD EDITION.

---

SOME time after the publication of an Octavo Edition of Euclid's Elements with Geometrical Exercises, &c., designed for the use of Academical Students; at the request of some schoolmasters of eminence, a duodecimo Edition of the Six Books was put forth on the same plan for the use of Schools. Soon after its appearance, Professor Christie, the Secretary of the Royal Society, in the Preface to his Treatise on Descriptive Geometry for the use of the Royal Military Academy, was pleased to notice these works in the following terms:—  
“When the greater Portion of this Part of the Course was printed, and had for some time been in use in the Academy, a new Edition of Euclid's Elements, by Mr. Robert Potts, M.A., of Trinity College, Cambridge, which is likely to supersede most others, to the extent, at least, of the Six Books, was published. From the manner of arranging the Demonstrations, this edition has the advantages of the symbolical form, and it is at the same time free from the manifold objections to which that form is open. The duodecimo edition of this Work, comprising only the first Six Books of Euclid, with Deductions from them, having been introduced at this Institution as a text-book, now renders any other Treatise on Plane Geometry unnecessary in our course of Mathematics.”

For the very favourable reception which both Editions have met with, the Editor's grateful acknowledgements are due. It has been his desire in putting forth a revised Edition of the School Euclid, to render the work in some degree more worthy of the favour which the former editions have received. In the present Edition several errors and oversights have been corrected and some additions made to the notes: the questions on each book have been considerably augmented and a better arrangement of the Geometrical Exercises has been attempted: and lastly, some hints and remarks on them have been given to assist the learner. The additions made to the present Edition amount to more than fifty pages, and, it is hoped, that they will render the work more useful to the learner.

And here an occasion may be taken to quote the opinions of some able men respecting the use and importance of the Mathematical Sciences.

On the subject of Education in its most extensive sense, an ancient writer “directs the aspirant after excellence to commence with the Science of Moral Culture; to proceed next to Logic; next to Mathematics; next to Physics; and lastly, to Theology.” Another writer on Education would place Mathematics before Logic, which (he remarks) “seems the preferable course: for by practising itself in the

former, the mind becomes stored with distinctions; the faculties of constancy and firmness are established; and its rule is always to distinguish between cavilling and investigation—between *close reasoning* and *cross reasoning*; for the contrary of all which habits, those are for the most part noted, who apply themselves to Logic without studying in some department of Mathematics; taking noise and wrangling for proficiency, and thinking refutation accomplished by the instancing of a doubt. This will explain the inscription placed by Plato over the door of his house: ‘Whoso knows not Geometry, let him not enter here.’ On the precedence of Moral Culture, however, to all the other Sciences, the acknowledgement is general, and the agreement entire.” The same writer recommends the study of the Mathematics, for the cure of “compound ignorance.” “Of this,” he proceeds to say, “the essence is opinion not agreeable to fact; and it necessarily involves another opinion, namely, that we are already possessed of knowledge. So that besides not knowing, we know not that we know not; and hence its designation of compound ignorance. In like manner, as of many chronic complaints and established maladies, no cure can be effected by physicians of the body: of this, no cure can be effected by physicians of the mind: for with a pre-supposal of knowledge in our own regard, the pursuit and acquirement of further knowledge is not to be looked for. The approximate cure, and one from which in the main much benefit may be anticipated, is to engage the patient in the study of measures (Geometry, computation, &c.); for in such pursuits the true and the false are separated by the clearest interval, and no room is left for the intrusions of fancy. From these the mind may discover the delight of certainty; and when, on returning to his own opinions, it finds in them no such sort of repose and gratification, it may discover their erroneous character, its ignorance may become simple, and a capacity for the acquirement of truth and virtue be obtained.”

Lord Bacon, the founder of Inductive Philosophy, was not insensible of the high importance of the Mathematical Sciences, as appears in the following passage from his work on “The Advancement of Learning.”

“The Mathematics are either pure or mixed. To the pure Mathematics are those sciences belonging which handle quantity determinate, merely severed from any axioms of natural philosophy; and these are two, Geometry, and Arithmetic; the one handling quantity continued, and the other dissevered. Mixed hath for subject some axioms or parts of natural philosophy, and considereth quantity determined, as it is auxiliary and incident unto them. For many parts of nature can-

neither be invented with sufficient subtlety, nor demonstrated with sufficient perspicuity, nor accommodated unto use with sufficient dexterity, without the aid and intervening of the Mathematics: of which sort are perspective, music, astronomy, cosmography, architecture, engineering, and divers others.

"In the Mathematics I can report no deficiency, except it be that men do not sufficiently understand the excellent use of the pure Mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For, if the wit be dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect that it maketh a quick eye, and a body ready to put itself into all postures; so in the Mathematics, that use which is collateral and intervenient, is no less worthy than that which is principal and intended. And as for the mixed Mathematics, I may only make this prediction, that there cannot fail to be more kinds of them, as nature grows further disclosed."

How truly has this prediction been fulfilled in the subsequent advancement of the Mixed Sciences, and in the applications of the pure Mathematics to Natural Philosophy!

Dr. Whewell, in his "Thoughts on the Study of Mathematics," has maintained, that mathematical studies judiciously pursued, form one of the most effective means of developing and cultivating the reason: and that "the object of a *liberal education* is to develop the whole mental system of man;—to make his speculative inferences coincide with his practical convictions;—to enable him to render a reason for the belief that is in him, and not to leave him in the condition of Solomon's sluggard, who is wiser in his own conceit than seven men that *can* render a reason." And in his more recent work entitled, "Of a Liberal Education, &c." he has more fully shewn the importance of Geometry as one of the most effectual instruments of intellectual education. In page 55 he thus proceeds:—"But besides the value of Mathematical Studies in Education, as a perfect example and complete exercise of demonstrative reasoning; Mathematical Truths have this additional recommendation, that they have always been referred to, by each successive generation of thoughtful and cultivated men, as examples of truth and of demonstration; and have thus become standard points of reference, among cultivated men, whenever they speak of truth, knowledge, or proof. Thus Mathematics has not only a disciplinary but an historical interest. This is peculiarly the case with those portions of Mathematics which we have mentioned. We find geometrical proof adduced in illustration of the

nature of reasoning, in the earliest speculations on this subject, the Dialogues of Plato; we find geometrical proof one of the main subjects of discussion in some of the most recent of such speculations, as those of Dugald Stewart and his contemporaries. The recollection of the truths of Elementary Geometry has, in all ages, given a meaning and a reality to the best attempts to explain man's power of arriving at truth. Other branches of Mathematics have, in like manner, become recognized examples, among educated men, of man's powers of attaining truth."

Dr. Pemberton, in the preface to his view of Sir Isaac Newton's Discoveries, makes mention of the circumstance, "that Newton used to speak with regret of his mistake, at the beginning of his Mathematical Studies, in having applied himself to the works of Descartes and other Algebraical writers, before he had considered the Elements of Euclid with the attention they deserve."

To these we may subjoin the opinion of Mr. John Stuart Mill, which he has recorded in his invaluable System of Logic, (Vol. II. p. 180) in the following terms. "The value of Mathematical instruction as a preparation for those more difficult investigations (physiology, society, government, &c.) consists in the applicability not of its doctrines, but of its method. Mathematics will ever remain the most perfect type of the Deductive Method in general; and the applications of Mathematics to the simpler branches of physics, furnish the only school in which philosophers can effectually learn the most difficult and important portion of their art, the employment of the laws of simpler phenomena for explaining and predicting those of the more complex. These grounds are quite sufficient for deeming mathematical training an indispensable basis of real scientific education, and regarding, with Plato, one who is *δυσεμπίτητος*, as wanting in one of the most essential qualifications for the successful cultivation of the higher branches of philosophy."

In addition to these authorities it may be remarked, that the new Regulations which were confirmed by a Grace of the Senate on the 11th of May, 1846, assign to Geometry and to Geometrical methods, a more important place in the Examinations both for Honors and for the Ordinary Degree in this University.

TRINITY COLLEGE,  
March 1, 1850.

R. P.

The supplement to the School Euclid (about forty-eight pages) has been incorporated with this impression of the Fifth Edition.

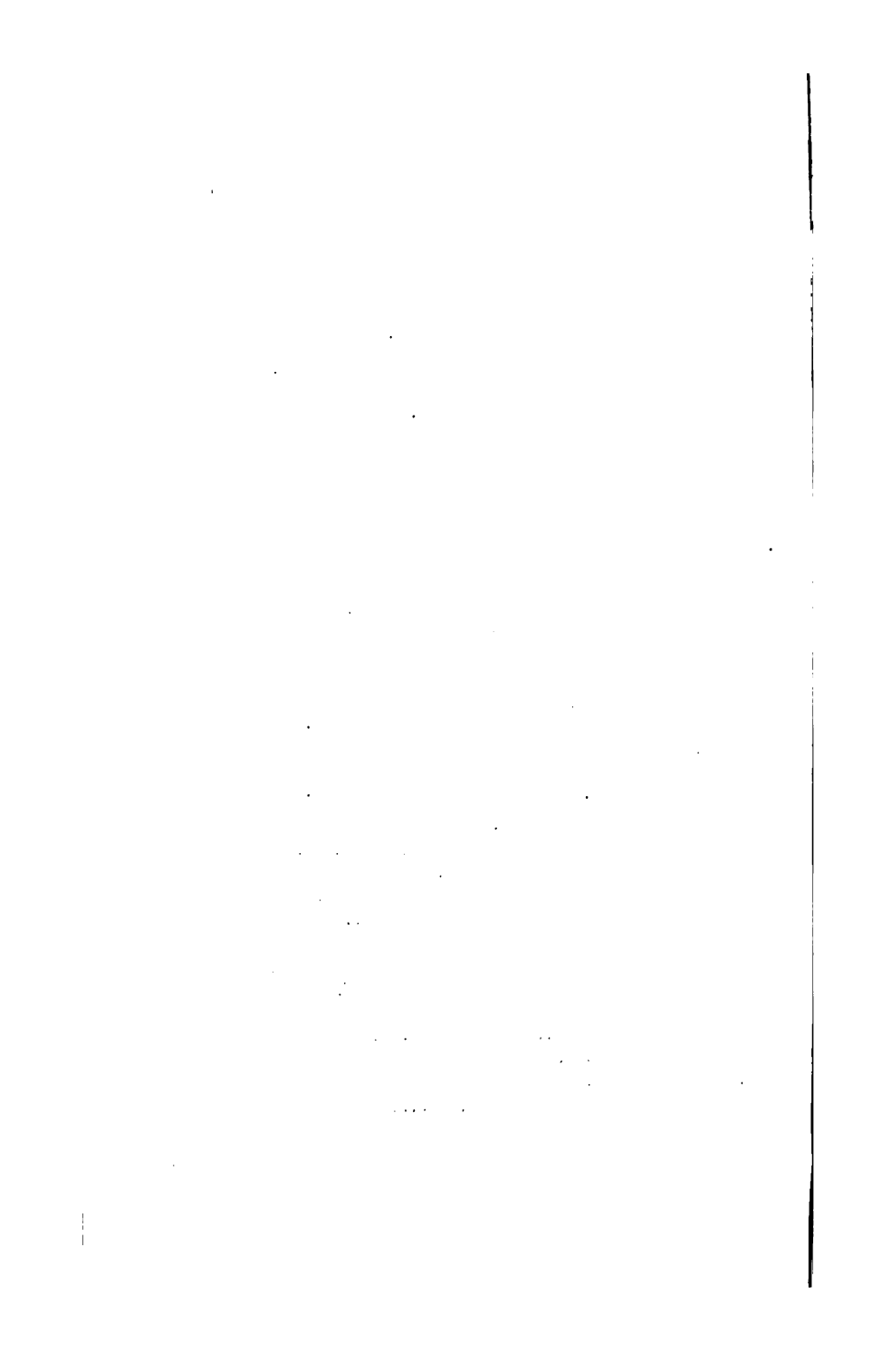
TRINITY COLLEGE,  
October, 1863.

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# EUCLID'S

## ELEMENTS OF GEOMETRY.

---

### BOOK I.

#### DEFINITIONS.

I.

A POINT is that which has no parts, or which has no magnitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A straight line is that which lies evenly between its extreme points.

V.

A superficies is that which has only length and breadth.

VI.

The extremities of a superficies are lines.

VII.

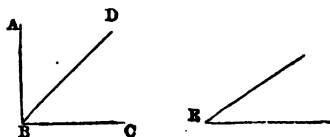
A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VIII.

A plane angle is the inclination of two lines to each other in a plane, which meet together, but are not in the same direction.

IX.

A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



N.B. If there be only one angle at a point, it may be expressed by a letter placed at that point, as the angle at  $E$ : but when several angles are at one point  $B$ , either of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of these straight lines, and the other upon the other line. Thus the angle which is contained by the straight lines  $AB$ ,  $CB$ , is named the angle  $ABC$ , or  $CBA$ ; that which is contained by  $AB$ ,  $DB$ , is named the angle  $ABD$ , or  $DBA$ ; and that which is contained by  $DB$ ,  $CB$ , is called the angle  $DBC$ , or  $CBD$ .

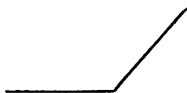
## X.

When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.



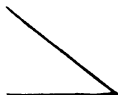
## XI.

An obtuse angle is that which is greater than a right angle.



## XII.

An acute angle is that which is less than a right angle.



## XIII.

A term or boundary is the extremity of any thing.

## XIV.

A figure is that which is enclosed by one or more boundaries.

## XV.

A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.

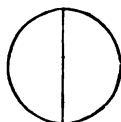


## XVI.

And this point is called the center of the circle.

## XVII.

A diameter of a circle is a straight line drawn through the center, and terminated both ways by the circumference.



## XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.



## XIX.

The center of a semicircle is the same with that of the circle.

## XX.

Rectilineal figures are those which are contained by straight lines.

## XXI.

Trilateral figures, or triangles, by three straight lines.

## XXII.

Quadrilateral, by four straight lines.

## XXIII.

Multilateral figures, or polygons, by more than four straight lines.

## XXIV.

Of three-sided figures, an equilateral triangle is that which has three equal sides.



## XXV.

An isosceles triangle is that which has two sides equal.



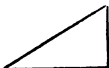
## XXVI.

A scalene triangle is that which has three unequal sides.



## XXVII.

A right-angled triangle is that which has a right angle.



## XXVIII.

An obtuse-angled triangle is that which has an obtuse angle.



## XXIX.

An acute-angled triangle is that which has three acute angles.



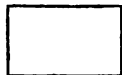
## XXX.

Of quadrilateral or four-sided figures, a square has all its sides equal and all its angles right angles.



## XXXI.

An oblong is that which has all its angles right angles, but has not all its sides equal.



## XXXII.

A rhombus has all its sides equal, but its angles are not right angles.



## XXXIII.

A rhomboid has its opposite sides equal to each other, but all its sides are not equal, nor its angles right angles.

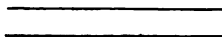


## XXXIV.

All other four-sided figures besides these, are called Trapeziums.

## XXXV.

Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways, do not meet.



## A.

A parallelogram is a four-sided figure, of which the opposite sides are parallel: and the diameter, or the diagonal is the straight line joining two of its opposite angles.

## POSTULATES.

## I.

LET it be granted that a straight line may be drawn from any one point to any other point.

## II.

That a terminated straight line may be produced to any length in a straight line.

## III.

And that a circle may be described from any center, at any distance from that center.

## AXIOMS.

## I.

THINGS which are equal to the same thing are equal to one another.

## II.

If equals be added to equals, the wholes are equal.

## III.

If equals be taken from equals, the remainders are equal.

## IV.

If equals be added to unequals, the wholes are unequal.

## V.

If equals be taken from unequals, the remainders are unequal.

## VI.

Things which are double of the same, are equal to one another.

## VII.

Things which are halves of the same, are equal to one another.

## VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

## IX.

The whole is greater than its part.

## X.

Two straight lines cannot enclose a space.

## XI.

All right angles are equal to one another.

## XII.

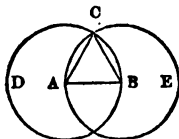
If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles; these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

PROPOSITION I. PROBLEM.

*To describe an equilateral triangle upon a given finite straight line.*

Let  $AB$  be the given straight line.

It is required to describe an equilateral triangle upon  $AB$ .



From the center  $A$ , at the distance  $AB$ , describe the circle  $BCD$ ;  
(post. 3.)

from the center  $B$ , at the distance  $BA$ , describe the circle  $ACE$ ;  
and from  $C$ , one of the points in which the circles cut one another,  
draw the straight lines  $CA, CB$  to the points  $A, B$ . (post. 1.)

Then  $ABC$  shall be an equilateral triangle.

Because the point  $A$  is the center of the circle  $BCD$ ,

therefore  $AC$  is equal to  $AB$ ; (def. 15.)

and because the point  $B$  is the center of the circle  $ACE$ ,

therefore  $BC$  is equal to  $AB$ ;

but it has been proved that  $AC$  is equal to  $AB$ ;

therefore  $AC, BC$  are each of them equal to  $AB$ ;

but things which are equal to the same thing are equal to one another;

therefore  $AC$  is equal to  $BC$ ; (ax. 1.)

wherefore  $AB, BC, CA$  are equal to one another:

and the triangle  $ABC$  is therefore equilateral,

and it is described upon the given straight line  $AB$ .

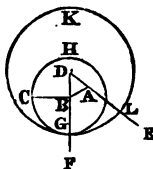
Which was required to be done.

PROPOSITION II. PROBLEM.

*From a given point, to draw a straight line equal to a given straight line.*

Let  $A$  be the given point, and  $BC$  the given straight line.

It is required to draw from the point  $A$ , a straight line equal to  $BC$ .



From the point  $A$  to  $B$  draw the straight line  $AB$ ; (post. 1.)

upon  $AB$  describe the equilateral triangle  $ABD$ , (i. 1.)

and produce the straight lines  $DA, DB$  to  $E$  and  $F$ ; (post. 2.)

from the center  $B$ , at the distance  $BC$ , describe the circle  $CGH$ ,  
(post. 3.) cutting  $DF$  in the point  $G$ ;

and from the center  $D$ , at the distance  $DG$ , describe the circle  $GKL$ ,  
cutting  $AE$  in the point  $L$ .



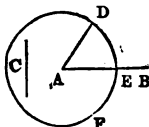
Then the straight line  $AL$  shall be equal to  $BC$ .  
 Because the point  $B$  is the center of the circle  $CGH$ ,  
 therefore  $BC$  is equal to  $BG$ ; (def. 15.)  
 and because  $D$  is the center of the circle  $GKL$ ,  
 therefore  $DL$  is equal to  $DG$ ,  
 and  $DA, DB$  parts of them are equal; (i. 1.)  
 therefore the remainder  $AL$  is equal to the remainder  $BG$ ; (ax. 3.)  
 but it has been shewn that  $BC$  is equal to  $BG$ ,  
 wherefore  $AL$  and  $BC$  are each of them equal to  $BG$ ;  
 and things that are equal to the same thing are equal to one another;  
 therefore the straight line  $AL$  is equal to  $BC$ . (ax. 1.)  
 Wherefore from the given point  $A$ , a straight line  $AL$  has been drawn  
 equal to the given straight line  $BC$ . Which was to be done.

## PROPOSITION III. PROBLEM.

*From the greater of two given straight lines to cut off a part equal to the less.*

Let  $AB$  and  $C$  be the two given straight lines, of which  $AB$  is the greater.

It is required to cut off from  $AB$  the greater, a part equal to  $C$ , the less.



From the point  $A$  draw the straight line  $AD$  equal to  $C$ ; (i. 2.)  
 and from the center  $A$ , at the distance  $AD$ , describe the circle  $DEF$   
 (post. 3.) cutting  $AB$  in the point  $E$ .

Then  $AE$  shall be equal to  $C$ .

Because  $A$  is the center of the circle  $DEF$ ,  
 therefore  $AE$  is equal to  $AD$ ; (def. 15.)

but the straight line  $C$  is equal to  $AD$ ; (constr.)

whence  $AE$  and  $C$  are each of them equal to  $AD$ ;

wherefore the straight line  $AE$  is equal to  $C$ . (ax. 1.)

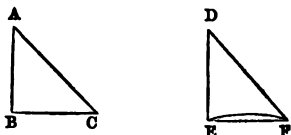
And therefore from  $AB$  the greater of two straight lines, a part  $AE$   
 has been cut off equal to  $C$ , the less. Which was to be done.

## PROPOSITION IV. THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to each other; they shall likewise have their bases or third sides equal, and the two triangles shall be equal, and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.*

Let  $ABC, DEF$  be two triangles, which have the two sides  $AB, AC$  equal to the two sides  $DE, DF$ , each to each, viz.  $AB$  to  $DE$ , and  $AC$  to  $DF$ , and the included angle  $BAC$  equal to the included angle  $EDF$ .

Then shall the base  $BC$  be equal to the base  $EF$ ; and the triangle  $ABC$  to the triangle  $DEF$ ; and the other angles to which the equal sides are opposite shall be equal, each to each, viz. the angle  $ABC$  to the angle  $DEF$ ; and the angle  $ACB$  to the angle  $DFE$ .



For, if the triangle  $ABC$  be applied to the triangle  $DEF$ , so that the point  $A$  may be on  $D$ , and the straight line  $AB$  on  $DE$ ; then the point  $B$  shall coincide with the point  $E$ , because  $AB$  is equal to  $DE$ ; and  $AB$  coinciding with  $DE$ , the straight line  $AC$  shall fall on  $DF$ , because the angle  $BAC$  is equal to the angle  $EDF$ ; therefore also the point  $C$  shall coincide with the point  $F$ , because  $AC$  is equal to  $DF$ ; but the point  $B$  was shewn to coincide with the point  $E$ ; wherefore the base  $BC$  shall coincide with the base  $EF$ ; because the point  $B$  coinciding with  $E$ , and  $C$  with  $F$ , if the base  $BC$  do not coincide with the base  $EF$ , the two straight lines  $BC$  and  $EF$  would enclose a space, which is impossible. (ax. 10.) Therefore the base  $BC$  does coincide with  $EF$ , and is equal to it; and the whole triangle  $ABC$  coincides with the whole triangle  $DEF$ , and is equal to it; also the remaining angles of one triangle coincide with the remaining angles of the other, and are equal to them, viz. the angle  $ABC$  to the angle  $DEF$ , and the angle  $ACB$  to  $DFE$ . Therefore, if two triangles have two sides of the one equal to two sides, &c. Which was to be demonstrated.

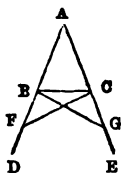
#### PROPOSITION V. THEOREM.

*The angles at the base of an isosceles triangle are equal to each other; and if the equal sides be produced, the angles on the other side of the base shall be equal.*

Let  $ABC$  be an isosceles triangle of which the side  $AB$  is equal to  $AC$ , and let the equal sides  $AB$ ,  $AC$  be produced to  $D$  and  $E$ . Then the angle  $ABC$  shall be equal to the angle  $ACB$ , and the angle  $DBC$  to the angle  $ECB$ .

In  $BD$  take any point  $F$ ; from  $AE$  the greater, cut off  $AG$  equal to  $AF$  the less, (I. 3.) and join  $FC$ ,  $GB$ .

Because  $AF$  is equal to  $AG$ , (constr.) and  $AB$  to  $AC$ ; (hyp.) the two sides  $FA$ ,  $AC$  are equal to the two  $GA$ ,  $AB$ , each to each; and they contain the angle  $FAG$  common to the two triangles  $AFC$ ,  $AGB$ ;



therefore the base  $FC$  is equal to the base  $GB$ , (I. 4.)

and the triangle  $AFC$  is equal to the triangle  $AGB$ ,  
also the remaining angles of the one are equal to the remaining angles  
of the other, each to each, to which the equal sides are opposite;  
viz. the angle  $ACF$  to the angle  $ABG$ ,  
and the angle  $AFC$  to the angle  $AGB$ .

And because the whole  $AF$  is equal to the whole  $AG$ ,  
of which the parts  $AB$ ,  $AC$ , are equal;

therefore the remainder  $BF$  is equal to the remainder  $CG$ ; (ax. 3.)

and  $FC$  has been proved to be equal to  $GB$ ;

hence, because the two sides  $BF$ ,  $FC$  are equal to the two  $CG$ ,  $GB$ ,  
each to each;

and the angle  $BFC$  has been proved to be equal to the angle  $CGB$ ,

also the base  $BC$  is common to the two triangles  $BFC$ ,  $CGB$ ;

wherefore these triangles are equal, (I. 4.)

and their remaining angles, each to each, to which the equal sides  
are opposite;

therefore the angle  $FBC$  is equal to the angle  $GCB$ ,

and the angle  $BCF$  to the angle  $CBG$ .

And, since it has been demonstrated,

that the whole angle  $ABG$  is equal to the whole  $ACF$ ,

the parts of which, the angles  $CBG$ ,  $BCF$  are also equal;

therefore the remaining angle  $ABC$  is equal to the remaining angle  $ACB$ ,

which are the angles at the base of the triangle  $ABC$ ;

and it has also been proved,

that the angle  $FBC$  is equal to the angle  $GCB$ ,

which are the angles upon the other side of the base.

Therefore the angles at the base, &c. Q.E.D.

COR. Hence an equilateral triangle is also equiangular.

#### PROPOSITION VI. THEOREM.

*If two angles of a triangle be equal to each other; the sides also which  
subtend, or are opposite to, the equal angles, shall be equal to one another.*

Let  $ABC$  be a triangle having the angle  $ABC$  equal to the angle  $ACB$ .

Then the side  $AB$  shall be equal to the side  $AC$ .



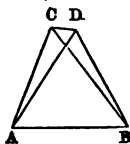
For, if  $AB$  be not equal to  $AC$ ,  
 one of them is greater than the other,  
 If possible, let  $AB$  be greater than  $AC$ ;  
 and from  $BA$  cut off  $BD$  equal to  $CA$  the less, (I. 3.) and join  $DC$ .  
 Then, in the triangles  $DBC$ ,  $ABC$ ,  
 because  $DB$  is equal to  $AC$ , and  $BC$  is common to both triangles,  
 the two sides  $DB$ ,  $BC$  are equal to the two sides  $AC$ ,  $CB$ , each to each;  
 and the angle  $DBC$  is equal to the angle  $ACB$ ; (hyp.)  
 therefore the base  $DC$  is equal to the base  $AB$ , (I. 4.)  
 and the triangle  $DBC$  is equal to the triangle  $ABC$ ,  
 the less equal to the greater, which is absurd. (ax. 9.)  
 Therefore  $AB$  is not unequal to  $AC$ , that is,  $AB$  is equal to  $AC$ .  
 Wherefore, if two angles, &c. Q.E.D.

COR. Hence an equiangular triangle is also equilateral.

### PROPOSITION VII. THEOREM.

*Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.*

If it be possible, on the same base  $AB$ , and upon the same side of it, let there be two triangles  $ACB$ ,  $ADB$ , which have their sides  $CA$ ,  $DA$ , terminated in the extremity  $A$  of the base, equal to one another, and likewise their sides  $CB$ ,  $DB$ , that are terminated in  $B$ .

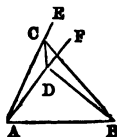


Join  $CD$ .

First. When the vertex of each of the triangles is without the other triangle.

Because  $AC$  is equal to  $AD$  in the triangle  $ACD$ ,  
 therefore the angle  $ADC$  is equal to the angle  $ACD$ ; (I. 5.)  
 but the angle  $ACD$  is greater than the angle  $BCD$ ; (ax. 9.)  
 therefore also the angle  $ADC$  is greater than  $BCD$ ;  
 much more therefore is the angle  $BDC$  greater than  $BCD$ .  
 Again, because the side  $BC$  is equal to  $BD$  in the triangle  $BCD$ , (hyp.)  
 therefore the angle  $BDC$  is equal to the angle  $BCD$ ; (I. 5.)  
 but the angle  $BDC$  was proved greater than the angle  $BCD$ ,  
 hence the angle  $BDC$  is both equal to, and greater than the angle  $BCD$ ;  
 which is impossible.

Secondly. Let the vertex  $D$  of the triangle  $ADB$  fall within the triangle  $ACB$ .



Produce  $AC$  to  $E$ , and  $AD$  to  $F$ , and join  $CD$ .

Then because  $AC$  is equal to  $AD$  in the triangle  $ACD$ , therefore the angles  $ECD$ ,  $FDC$  upon the other side of the base  $CD$ , are equal to one another; (1. 5.)

but the angle  $ECD$  is greater than the angle  $BCD$ ; (ax. 9.) therefore also the angle  $FDC$  is greater than the angle  $BCD$ ; much more then is the angle  $BDC$  greater than the angle  $BCD$ .

Again, because  $BC$  is equal to  $BD$  in the triangle  $BCD$ , therefore the angle  $BDC$  is equal to the angle  $BCD$ . (1. 5.)

but the angle  $BDC$  has been proved greater than  $BCD$ , wherefore the angle  $BDC$  is both equal to, and greater than the angle  $BCD$ ; which is impossible.

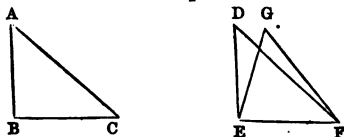
Thirdly. The case in which the vertex of one triangle is upon a side of the other, needs no demonstration.

Therefore, upon the same base and on the same side of it, &c. Q.E.D.

### PROPOSITION VIII. THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.*

Let  $ABC$ ,  $DEF$  be two triangles, having the two sides  $AB$ ,  $AC$ , equal to the two sides  $DE$ ,  $DF$ , each to each, viz.  $AB$  to  $DE$ , and  $AC$  to  $DF$ , and also the base  $BC$  equal to the base  $EF$ .



Then the angle  $BAC$  shall be equal to the angle  $EDF$ .

For, if the triangle  $ABC$  be applied to  $DEF$ , so that the point  $B$  be on  $E$ , and the straight line  $BC$  on  $EF$ ;

then because  $BC$  is equal to  $EF$ , (hyp.)

therefore the point  $C$  shall coincide with the point  $F$ ;

wherefore  $BC$  coinciding with  $EF$ ,

$BA$  and  $AC$  shall coincide with  $ED$ ,  $DF$ ;

for, if the base  $BC$  coincide with the base  $EF$ , but the sides  $BA$ ,  $AC$ , do not coincide with the sides  $ED$ ,  $DF$ , but have a different situation as  $EG$ ,  $GF$ :

then, upon the same base, and upon the same side of it, there can be two triangles which have their sides which are terminated in one extremity of the base, equal to one another, and likewise those sides which are terminated in the other extremity; but this is impossible. (1. 7.)

Therefore, if the base  $BC$  coincide with the base  $EF$ ,

the sides  $BA$ ,  $AC$  cannot but coincide with the sides  $ED$ ,  $DF$ ; wherefore likewise the angle  $BAC$  coincides with the angle  $EDF$ , and is equal to it. (ax. 8.)

Therefore if two triangles have two sides, &c. Q.E.D.

## PROPOSITION IX. PROBLEM.

*To bisect a given rectilineal angle, that is, to divide it into two equal angles.*

Let  $BAC$  be the given rectilineal angle.

It is required to bisect it.



In  $AB$  take any point  $D$ ;  
from  $AC$  cut off  $AE$  equal to  $AD$ , (I. 3.) and join  $DE$ ;  
on the side of  $DE$  remote from  $A$ ,  
describe the equilateral triangle  $DEF$  (I. 1.), and join  $AF$ .

Then the straight line  $AF$  shall bisect the angle  $BAC$ .

Because  $AD$  is equal to  $AE$ , (constr.)

and  $AF$  is common to the two triangles  $DAF$ ,  $EAF$ ;  
the two sides  $DA$ ,  $AF$ , are equal to the two sides  $EA$ ,  $AF$ , each to each;

and the base  $DF$  is equal to the base  $EF$ : (constr.)

therefore the angle  $DAF$  is equal to the angle  $EAF$ . (I. 8.)

Wherefore the angle  $BAC$  is bisected by the straight line  $AF$ . Q.E.F.

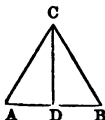
## PROPOSITION X. PROBLEM.

*To bisect a given finite straight line, that is, to divide it into two equal parts.*

Let  $AB$  be the given straight line.

It is required to divide  $AB$  into two equal parts.

Upon  $AB$  describe the equilateral triangle  $ABC$ ; (I. 1.)



and bisect the angle  $ACB$  by the straight line  $CD$  meeting  $AB$  in the point  $D$ . (I. 9.)

Then  $AB$  shall be cut into two equal parts in the point  $D$ .

Because  $AC$  is equal to  $CB$ , (constr.)

and  $CD$  is common to the two triangles  $ACD$ ,  $BCD$ ;

the two sides  $AC$ ,  $CD$  are equal to the two  $BC$ ,  $CD$ , each to each;

and the angle  $ACD$  is equal to  $BCD$ ; (constr.)

therefore the base  $AD$  is equal to the base  $BD$ . (I. 4.)

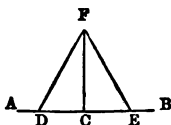
Wherefore the straight line  $AB$  is divided into two equal parts in the point  $D$ . Q.E.F.

## PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let  $AB$  be the given straight line, and  $C$  a given point in it.

It is required to draw a straight line from the point  $C$  at right angles to  $AB$ .



In  $AC$  take any point  $D$ , and make  $CE$  equal to  $CD$ ; (I. 3.)  
upon  $DE$  describe the equilateral triangle  $DEF$  (I. 1), and join  $CF$ .

Then  $CF$  drawn from the point  $C$ , shall be at right angles to  $AB$ .

Because  $DC$  is equal to  $EC$ , and  $FC$  is common to the two triangles  $DCF$ ,  $ECF$ ;

the two sides  $DC$ ,  $CF$  are equal to the two sides  $EC$ ,  $CF$ , each to each;  
and the base  $DF$  is equal to the base  $EF$ ; (constr.)

therefore the angle  $DCF$  is equal to the angle  $ECF$ : (I. 8.)

and these two angles are adjacent angles.

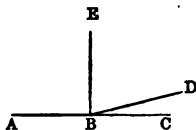
But when the two adjacent angles which one straight line makes with another straight line, are equal to one another, each of them is called a right angle: (def. 10.)

therefore each of the angles  $DCF$ ,  $ECF$  is a right angle.

Wherefore from the given point  $C$ , in the given straight line  $AB$ ,  $FC$  has been drawn at right angles to  $AB$ . Q.E.F.

COR. By help of this problem, it may be demonstrated that two straight lines cannot have a common segment.

If it be possible, let the segment  $AB$  be common to the two straight lines  $ABC$ ,  $ABD$ .



From the point  $B$ , draw  $BE$  at right angles to  $AB$ ; (I. 11.)

then because  $ABC$  is a straight line,

therefore the angle  $ABE$  is equal to the angle  $EBC$ . (def. 10.)

Similarly, because  $ABD$  is a straight line,

therefore the angle  $ABE$  is equal to the angle  $EBD$ ;

but the angle  $ABE$  is equal to the angle  $EBC$ ,

wherefore the angle  $EBD$  is equal to the angle  $EBC$ , (ax. 1.)

the less equal to the greater angle, which is impossible.

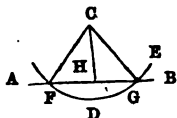
Therefore two straight lines cannot have a common segment.

## PROPOSITION XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let  $AB$  be the given straight line, which may be produced any length both ways, and let  $C$  be a point without it.

It is required to draw a straight line perpendicular to  $AB$  from the point  $C$ .



Upon the other side of  $AB$  take any point  $D$ , and from the center  $C$ , at the distance  $CD$ , describe the circle  $EGF$  meeting  $AB$ , produced if necessary, in  $F$  and  $G$ : (post. 3.) bisect  $FG$  in  $H$  (I. 10.), and join  $CH$ .

Then the straight line  $CH$  drawn from the given point  $C$ , shall be perpendicular to the given straight line  $AB$ .

Join  $FC$ , and  $CG$ .

Because  $FH$  is equal to  $HG$ , (constr.)

and  $HC$  is common to the triangles  $FHC$ ,  $GHC$ ;

the two sides  $FH$ ,  $HC$ , are equal to the two  $GH$ ,  $HC$ , each to each;

and the base  $CF$  is equal to the base  $CG$ ; (def. 15.)

therefore the angle  $FHC$  is equal to the angle  $GHC$ ; (I. 8.)

and these are adjacent angles.

But when a straight line standing on another straight line, makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other is called a perpendicular to it. (def. 10.)

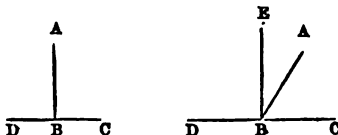
Therefore from the given point  $C$ , a perpendicular  $CH$  has been drawn to the given straight line  $AB$ . Q.E.F.

### PROPOSITION XIII. THEOREM.

*The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.*

Let the straight line  $AB$  make with  $CD$ , upon one side of it, the angles  $CBA$ ,  $ABD$ .

Then these shall be either two right angles,  
or, shall be together, equal to two right angles.



For if the angle  $CBA$  be equal to the angle  $ABD$ ,  
each of them is a right angle. (def. 10.)

But if the angle  $CBA$  be not equal to the angle  $ABD$ ,  
from the point  $B$  draw  $BE$  at right angles to  $CD$ . (I. 11.)  
Then the angles  $CBE$ ,  $EBD$  are two right angles. (def. 10.)



And because the angle  $CBE$  is equal to the angles  $CBA, ABE$ ,  
add the angle  $EBD$  to each of these equals;  
therefore the angles  $CBE, EBD$  are equal to the three angles  $CBA,$   
 $ABE, EBD$ . (ax. 2.)

Again, because the angle  $DBA$  is equal to the two angles  $DBE, EBA$ ,  
add to each of these equals the angle  $ABC$ ;  
therefore the angles  $DBA, ABC$  are equal to the three angles  $DBE,$   
 $EBA, ABC$ .

But the angles  $CBE, EBD$  have been proved equal to the same  
three angles;  
and things which are equal to the same thing are equal to one another;  
therefore the angles  $CBE, EBD$  are equal to the angles  $DBA, ABC$ ;  
but the angles  $CBE, EBD$  are two right angles;  
therefore the angles  $DBA, ABC$  are together equal to two right angles.  
(ax. 1.)

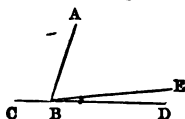
Wherefore, when a straight line, &c. Q.E.D.

#### PROPOSITION XIV. THEOREM.

*If at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles; then these two straight lines shall be in one and the same straight line.*

At the point  $B$  in the straight line  $AB$ , let the two straight lines  
 $BC, BD$  upon the opposite sides of  $AB$ , make the adjacent angles  
 $ABC, ABD$  together equal to two right angles.

Then  $BD$  shall be in the same straight line with  $BC$ .



For, if  $BD$  be not in the same straight line with  $BC$ ,  
if possible, let  $BE$  be in the same straight line with it.

Then because  $AB$  meets the straight line  $CBE$ ;  
therefore the adjacent angles  $CBA, ABE$  are equal to two right angles;  
(I. 13.)

but the angles  $CBA, ABD$  are equal to two right angles; (hyp.)  
therefore the angles  $CBA, ABE$  are equal to the angles  $CBA, ABD$ ;  
(ax. 1.)

take away from these equals the common angle  $CBA$ ,  
therefore the remaining angle  $ABE$  is equal to the remaining angle  
 $ABD$ ; (ax. 3.)

the less angle equal to the greater, which is impossible:  
therefore  $BE$  is not in the same straight line with  $BC$ .

And in the same manner it may be demonstrated, that no other  
can be in the same straight line with it but  $BD$ , which therefore is in  
the same straight line with  $BC$ .

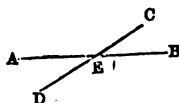
Wherefore, if at a point, &c. Q.E.D.

## PROPOSITION XV. THEOREM.

*If two straight lines cut one another, the vertical, or opposite angles shall be equal.*

Let the two straight lines  $AB$ ,  $CD$  cut one another in the point  $E$ .

Then the angle  $AEC$  shall be equal to the angle  $DEB$ , and the angle  $CEB$  to the angle  $AED$ .



Because the straight line  $AE$  makes with  $CD$  at the point  $E$ , the adjacent angles  $CEA$ ,  $AED$ ;

these angles are together equal to two right angles. (I. 13.)

Again, because the straight line  $DE$  makes with  $AB$  at the point  $E$ , the adjacent angles  $AED$ ,  $DEB$ ;

these angles also are equal to two right angles;

but the angles  $CEA$ ,  $AED$  have been shewn to be equal to two right angles;

wherefore the angles  $CEA$ ,  $AED$  are equal to the angles  $AED$ ,  $DEB$ ;

take away from each the common angle  $AED$ ,

and the remaining angle  $CEA$  is equal to the remaining angle  $DEB$ . (ax. 3.)

In the same manner it may be demonstrated, that the angle  $CEB$  is equal to the angle  $AED$ .

Therefore, if two straight lines cut one another, &c. Q. E. D.

COR. 1. From this it is manifest, that, if two straight lines cut each other, the angles which they make at the point where they cut, are together equal to four right angles.

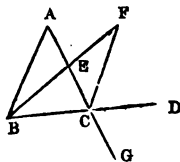
COR. 2. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

## PROPOSITION XVI. THEOREM.

*If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.*

Let  $ABC$  be a triangle, and let the side  $BC$  be produced to  $D$ .

Then the exterior angle  $ACD$  shall be greater than either of the interior opposite angles  $CBA$  or  $BAC$ .



Bisect  $AC$  in  $E$ , (I. 10.) and join  $BE$ ;  
produce  $BE$  to  $F$ , making  $EF$  equal to  $BE$ , (I. 3.) and join  $FC$ .

Because  $AE$  is equal to  $EC$ , and  $BE$  to  $EF$ ; (constr.)  
the two sides  $AE$ ,  $EB$  are equal to the two  $CE$ ,  $EF$ , each to each, in  
the triangles  $ABE$ ,  $CFE$ ;

and the angle  $AEB$  is equal to the angle  $CEF$ ,  
because they are opposite vertical angles; (I. 15.)

therefore the base  $AB$  is equal to the base  $CF$ , (I. 4.)

and the triangle  $ABE$  to the triangle  $CFE$ ,

and the remaining angles of one triangle to the remaining angles of  
the other, each to each, to which the equal sides are opposite;

wherefore the angle  $BAE$  is equal to the angle  $ECF$ ;

but the angle  $ECD$  or  $ACD$  is greater than the angle  $ECF$ ;

therefore the angle  $ACD$  is greater than the angle  $BAE$  or  $BAC$ .

In the same manner, if the side  $BC$  be bisected, and  $AC$  be pro-  
duced to  $G$ ; it may be demonstrated that the angle  $BCG$ , that is, the  
angle  $ACD$ , (I. 15.) is greater than the angle  $ABC$ .

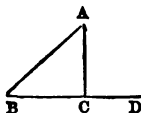
Therefore, if one side of a triangle, &c. Q.E.D.

#### PROPOSITION XVII. THEOREM.

*Any two angles of a triangle are together less than two right angles.*

Let  $ABC$  be any triangle.

Then any two of its angles together shall be less than two right angles.



Produce any side  $BC$  to  $D$ .

Then because  $ACD$  is the exterior angle of the triangle  $ABC$ ;  
therefore the angle  $ACD$  is greater than the interior and opposite angle  
 $ABC$ ; (I. 16.)

to each of these unequals add the angle  $ACB$ ;

therefore the angles  $ACD$ ,  $ACB$  are greater than the angles  $ABC$ ,  
 $ACB$ ;

but the angles  $ACD$ ,  $ACB$  are equal to two right angles; (I. 13.)  
therefore the angles  $ABC$ ,  $ACB$  are less than two right angles.

In like manner it may be demonstrated,

that the angles  $BAC$ ,  $ACB$  are less than two right angles,  
as also the angles  $CAB$ ,  $ABC$ .

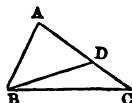
Therefore any two angles of a triangle, &c. Q.E.D.

#### PROPOSITION XVIII. THEOREM.

*The greater side of every triangle is opposite to the greater angle.*

Let  $ABC$  be a triangle, of which the side  $AC$  is greater than the  
side  $AB$ .

Then the angle  $ABC$  shall be greater than the angle  $ACB$ .



Since the side  $AC$  is greater than the side  $AB$ , (hyp.)  
 make  $AD$  equal to  $AB$ , (I. 3.) and join  $BD$ .  
 Then, because  $AD$  is equal to  $AB$ , in the triangle  $ABD$ ,  
 therefore the angle  $ABD$  is equal to the angle  $ADB$ , (I. 5.)  
 but because the side  $CD$  of the triangle  $BDC$  is produced to  $A$ ,  
 therefore the exterior angle  $ADB$  is greater than the interior and  
 opposite angle  $DCB$ ; (I. 16.)  
 but the angle  $ADB$  has been proved equal to the angle  $ABD$ ,  
 therefore the angle  $ABD$  is greater than the angle  $DCB$ ;  
 wherefore much more is the angle  $ABC$  greater than the angle  $ACB$ .  
 Therefore the greater side, &c. Q. E. D.

#### PROPOSITION XIX. THEOREM.

*The greater angle of every triangle is subtended by the greater side, or, has the greater side opposite to it.*

Let  $ABC$  be a triangle of which the angle  $ABC$  is greater than the angle  $BCA$ .

Then the side  $AC$  shall be greater than the side  $AB$ .



For, if  $AC$  be not greater than  $AB$ ,  
 $AC$  must either be equal to, or less than  $AB$ ;  
 if  $AC$  were equal to  $AB$ ,  
 then the angle  $ABC$  would be equal to the angle  $ACB$ ; (I. 6.)  
 but it is not equal; (hyp.)  
 therefore the side  $AC$  is not equal to  $AB$ .  
 Again, if  $AC$  were less than  $AB$ ,  
 then the angle  $ABC$  would be less than the angle  $ACB$ ; (I. 18.)  
 but it is not less, (hyp.)  
 therefore the side  $AC$  is not less than  $AB$ ;  
 and  $AC$  has been shewn to be not equal to  $AB$ ;  
 therefore  $AC$  is greater than  $AB$ .  
 Wherefore the greater angle, &c. Q. E. D.

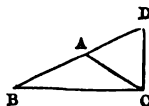
#### PROPOSITION XX. THEOREM.

*Any two sides of a triangle are together greater than the third side.*

Let  $ABC$  be a triangle.

Then any two sides of it together shall be greater than the third side,  
 viz. the sides  $BA$ ,  $AC$  greater than the side  $BC$ ;

$AB, BC$  greater than  $AC$ ;  
and  $BC, CA$  greater than  $AB$ .



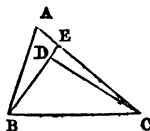
Produce the side  $BA$  to the point  $D$ ,  
make  $AD$  equal to  $AC$ , (I. 3.) and join  $DC$ .  
Then because  $AD$  is equal to  $AC$ , (constr.)  
therefore the angle  $ACD$  is equal to the angle  $ADC$ ; (I. 5.)  
but the angle  $BCD$  is greater than the angle  $ACD$ ; (ax. 9.)  
therefore also the angle  $BCD$  is greater than the angle  $ADC$ .  
And because in the triangle  $DBC$ ,  
the angle  $BCD$  is greater than the angle  $BDC$ ,  
and that the greater angle is subtended by the greater side; (I. 19.)  
therefore the side  $DB$  is greater than the side  $BC$ ;  
but  $DB$  is equal to  $BA$  and  $AC$ ,  
therefore the sides  $BA$  and  $AC$  are greater than  $BC$ .  
In the same manner it may be demonstrated,  
that the sides  $AB, BC$  are greater than  $CA$ ;  
also that  $BC, CA$  are greater than  $AB$ .  
Therefore any two sides, &c. Q. E. D.

### PROPOSITION XXI. THEOREM.

*If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle; these shall be less than the other two sides of the triangle, but shall contain a greater angle.*

Let  $ABC$  be a triangle, and from the points  $B, C$ , the ends of the side  $BC$ , let the two straight lines  $BD, CD$  be drawn to a point  $D$  within the triangle.

Then  $BD$  and  $DC$  shall be less than  $BA$  and  $AC$  the other two sides of the triangle,  
but shall contain an angle  $BDC$  greater than the angle  $BAC$ .



Produce  $BD$  to meet the side  $AC$  in  $E$ .  
Because two sides of a triangle are greater than the third side, (I. 20.)  
therefore the two sides  $BA, AE$  of the triangle  $ABE$  are greater than  $BE$ ;

to each of these unequals add  $EC$ ;  
therefore the sides  $BA, AC$  are greater than  $BE, EC$ . (ax. 4.)  
Again, because the two sides  $CE, ED$  of the triangle  $CED$  are greater than  $DC$ ; (I. 20.)

add  $DB$  to each of these unequals;

therefore the sides  $CE, EB$  are greater than  $CD, DB$ . (ax. 4.)

But it has been shewn that  $BA, AC$  are greater than  $BE, EC$ ;

much more then are  $BA, AC$  greater than  $BD, DC$ .

Again, because the exterior angle of a triangle is greater than the interior and opposite angle; (I. 16.)

therefore the exterior angle  $BDC$  of the triangle  $CDE$  is greater than the interior and opposite angle  $CED$ ;

for the same reason, the exterior angle  $CED$  of the triangle  $ABE$  is greater than the interior and opposite angle  $BAC$ ;

and it has been demonstrated,

that the angle  $BDC$  is greater than the angle  $CEB$ ;

much more therefore is the angle  $BDC$  greater than the angle  $BAC$ .

Therefore, if from the ends of the side, &c. Q. E. D.

### PROPOSITION XXII. PROBLEM.

*To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.*

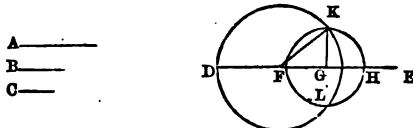
Let  $A, B, C$  be the three given straight lines,  
of which any two whatever are greater than the third, (I. 20.)

namely,  $A$  and  $B$  greater than  $C$ ;

$A$  and  $C$  greater than  $B$ ;

and  $B$  and  $C$  greater than  $A$ .

It is required to make a triangle of which the sides shall be equal to  $A, B, C$ , each to each.



Take a straight line  $DE$  terminated at the point  $D$ , but unlimited towards  $E$ ,

make  $DF$  equal to  $A$ ,  $FG$  equal to  $B$ , and  $GH$  equal to  $C$ ; (I. 3.)  
from the center  $F$ , at the distance  $FD$ , describe the circle  $DKL$ ;  
(post 3.)

from the center  $G$ , at the distance  $GH$ , describe the circle  $HLK$ ;  
from  $K$  where the circles cut each other, draw  $KF, KG$  to the points  $F, G$ ;

Then the triangle  $KFG$  shall have its sides equal to the three straight lines  $A, B, C$ .

Because the point  $F$  is the center of the circle  $DKL$ ,

therefore  $FD$  is equal to  $FK$ ; (def. 15.)

but  $FD$  is equal to the straight line  $A$ ;

therefore  $FK$  is equal to  $A$ .

Again, because  $G$  is the center of the circle  $HLK$ ;

therefore  $GH$  is equal to  $GK$ , (def. 15.)

but  $GH$  is equal to  $C$ ;

therefore also  $GK$  is equal to  $C$ ; (ax. 1.)

and  $FG$  is equal to  $B$ ;

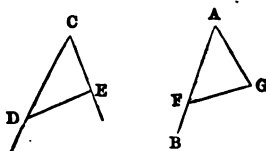
therefore the three straight lines  $KF$ ,  $FG$ ,  $GK$ , are respectively equal to the three,  $A$ ,  $B$ ,  $C$ :  
and therefore the triangle  $KFG$  has its three sides  $KF$ ,  $FG$ ,  $GK$ , equal to the three given straight lines  $A$ ,  $B$ ,  $C$ . Q.E.F.

### PROPOSITION XXIII. PROBLEM.

*At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.*

Let  $AB$  be the given straight line, and  $A$  the given point in it, and  $DCE$  the given rectilineal angle.

It is required, at the given point  $A$  in the given straight line  $AB$ , to make an angle that shall be equal to the given rectilineal angle  $DCE$ .



In  $CD$ ,  $CE$ , take any points  $D$ ,  $E$ , and join  $DE$ ;  
on  $AB$ , make the triangle  $FAG$ , the sides of which shall be equal to the three straight lines  $CD$ ,  $DE$ ,  $EC$ , so that  $AF$  be equal to  $CD$ ,  $AG$  to  $CE$ , and  $FG$  to  $DE$ . (I. 22.)

Then the angle  $FAG$  shall be equal to the angle  $DCE$ .

Because  $FA$ ,  $AG$  are equal to  $DC$ ,  $CE$ , each to each,  
and the base  $FG$  is equal to the base  $DE$ ;

therefore the angle  $FAG$  is equal to the angle  $DCE$ . (I. 8.)

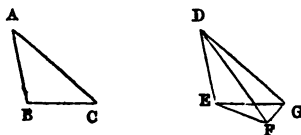
Wherefore, at the given point  $A$  in the given straight line  $AB$ , the angle  $FAG$  is made equal to the given rectilineal angle  $DCE$ . Q.E.F.

### PROPOSITION XXIV. THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle, shall be greater than the base of the other.*

Let  $ABC$ ,  $DEF$  be two triangles, which have the two sides  $AB$ ,  $AC$ , equal to the two  $DE$ ,  $DF$ , each to each, namely,  $AB$  equal to  $DE$ , and  $AC$  to  $DF$ ; but the angle  $BAC$  greater than the angle  $EDF$ .

Then the base  $BC$  shall be greater than the base  $EF$ .



Of the two sides  $DE$ ,  $DF$ , let  $DE$  be not greater than  $DF$ , at the point  $D$ , in the line  $DE$ , and on the same side of it as  $DF$ , make the angle  $EDG$  equal to the angle  $BAC$ ; (I. 23.) make  $DG$  equal to  $DF$  or  $AC$ , (I. 3.) and join  $EG$ ,  $GF$ .

Then, because  $DE$  is equal to  $AB$ , and  $DG$  to  $AC$ , the two sides  $DE$ ,  $DG$  are equal to the two  $AB$ ,  $AC$ , each to each, and the angle  $EDG$  is equal to the angle  $BAC$ ; therefore the base  $EG$  is equal to the base  $BC$ . (I. 4.)

And because  $DG$  is equal to  $DF$  in the triangle  $DFG$ , therefore the angle  $DFG$  is equal to the angle  $DGF$ ; (I. 5.) but the angle  $DGF$  is greater than the angle  $EGF$ ; (ax. 9.) therefore the angle  $DFG$  is also greater than the angle  $EGF$ ; much more therefore is the angle  $EFG$  greater than the angle  $EGF$ .

And because in the triangle  $EFG$ , the angle  $EFG$  is greater than the angle  $EGF$ ,

and that the greater angle is subtended by the greater side; (I. 19.)

therefore the side  $EG$  is greater than the side  $EF$ ;

but  $EG$  was proved equal to  $BC$ ;

therefore  $BC$  is greater than  $EF$ .

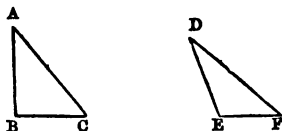
Wherefore, if two triangles, &c. Q. E. D.

#### PROPOSITION XXV. THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; the angle contained by the sides of the one which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.*

Let  $ABC$ ,  $DEF$  be two triangles which have the two sides  $AB$ ,  $AC$ , equal to the two sides  $DE$ ,  $DF$ , each to each, namely,  $AB$  equal to  $DE$ , and  $AC$  to  $DF$ ; but the base  $BC$  greater than the base  $EF$ .

Then the angle  $BAC$  shall be greater than the angle  $EDF$ .



For, if the angle  $BAC$  be not greater than the angle  $EDF$ , it must either be equal to it, or less than it.

If the angle  $BAC$  were equal to the angle  $EDF$ , then the base  $BC$  would be equal to the base  $EF$ ; (I. 4.) but it is not equal, (hyp.)

therefore the angle  $BAC$  is not equal to the angle  $EDF$ .

Again, if the angle  $BAC$  were less than the angle  $EDF$ , then the base  $BC$  would be less than the base  $EF$ ; (I. 24.)

but it is not less, (hyp.)

therefore the angle  $BAC$  is not less than the angle  $EDF$ ; and it has been shewn, that the angle  $BAC$  is not equal to the angle  $EDF$ ;

therefore the angle  $BAC$  is greater than the angle  $EDF$ .

Wherefore, if two triangles, &c. Q. E. D.



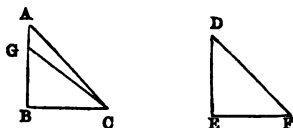
## PROPOSITION XXVI. THEOREM.

*If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz. either the sides adjacent to the equal angles in each, or the sides opposite to them; then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.*

Let  $ABC$ ,  $DEF$  be two triangles which have the angles  $ABC$ ,  $BCA$ , equal to the angles  $DEF$ ,  $EFD$ , each to each, namely,  $ABC$  to  $DEF$ , and  $BCA$  to  $EFD$ ; also one side equal to one side.

First, let those sides be equal which are adjacent to the angles that are equal in the two triangles, namely,  $BC$  to  $EF$ .

Then the other sides shall be equal, each to each, namely,  $AB$  to  $DE$ , and  $AC$  to  $DF$ , and the third angle  $BAC$  to the third angle  $EDF$ .



For, if  $AB$  be not equal to  $DE$ , one of them must be greater than the other.

If possible, let  $AB$  be greater than  $DE$ , make  $BG$  equal to  $DE$ , (I. 3.) and join  $GC$ .

Then in the two triangles  $GBC$ ,  $DEF$ , because  $GB$  is equal to  $DE$ , and  $BC$  to  $EF$ , (hyp.) the two sides  $GB$ ,  $BC$  are equal to the two  $DE$ ,  $EF$ , each to each; and the angle  $GBC$  is equal to the angle  $DEF$ ; therefore the base  $GC$  is equal to the base  $DF$ , (I. 4.) and the triangle  $GBC$  to the triangle  $DEF$ ,

and the other angles to the other angles, each to each, to which the equal sides are opposite;

therefore the angle  $GCB$  is equal to the angle  $DFE$ ; but the angle  $ACB$  is, by the hypothesis, equal to the angle  $DFE$ ; wherefore also the angle  $GCB$  is equal to the angle  $ACB$ ; (ax. 1.) the less angle equal to the greater, which is impossible;

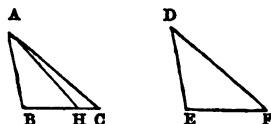
therefore  $AB$  is not unequal to  $DE$ , that is,  $AB$  is equal to  $DE$ .

Hence, in the triangles  $ABC$ ,  $DEF$ ;

because  $AB$  is equal to  $DE$ , and  $BC$  to  $EF$ , (hyp.) and the angle  $ABC$  is equal to the angle  $DEF$ ; (hyp.) therefore the base  $AC$  is equal to the base  $DF$ , (I. 4.) and the third angle  $BAC$  to the third angle  $EDF$ .

Secondly, let the sides which are opposite to one of the equal angles in each triangle be equal to one another, namely,  $AB$  equal to  $DE$ .

Then in this case likewise the other sides shall be equal,  $AC$  to  $DF$ , and  $BC$  to  $EF$ , and also the third angle  $BAC$  to the third angle  $EDF$ .



For if  $BC$  be not equal to  $EF$ ,  
one of them must be greater than the other.

If possible, let  $BC$  be greater than  $EF$ ;  
make  $BH$  equal to  $EF$ , (I. 3.) and join  $AH$ .

Then in the two triangles  $ABH$ ,  $DEF$ ,  
because  $AB$  is equal to  $DE$ , and  $BH$  to  $EF$ ,  
and the angle  $ABH$  to the angle  $DEF$ ; (hyp.)  
therefore the base  $AH$  is equal to the base  $DF$ , (I. 4.)  
and the triangle  $ABH$  to the triangle  $DEF$ ,

and the other angles to the other angles, each to each, to which the  
equal sides are opposite;

therefore the angle  $BHA$  is equal to the angle  $EFD$ ;  
but the angle  $EFD$  is equal to the angle  $BCA$ ; (hyp.)  
therefore the angle  $BHA$  is equal to the angle  $BCA$ , (ax. 1.)  
that is, the exterior angle  $BHA$  of the triangle  $AHC$ , is  
equal to its interior and opposite angle  $BCA$ ;

which is impossible; (I. 16.)  
wherefore  $BC$  is not unequal to  $EF$ ,  
that is,  $BC$  is equal to  $EF$ .

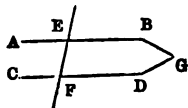
Hence, in the triangles  $ABC$ ,  $DEF$ ;  
because  $AB$  is equal to  $DE$ , and  $BC$  to  $EF$ , (hyp.)  
and the included angle  $ABC$  is equal to the included angle  $DEF$ ; (hyp.)  
therefore the base  $AC$  is equal to the base  $DF$ , (I. 4.)  
and the third angle  $BAC$  to the third angle  $EDF$ .

Wherefore, if two triangles, &c. Q.E.D.

# PROPOSITION XXVII. THEOREM.

*If a straight line falling on two other straight lines, make the alternate angles equal to each other; these two straight lines shall be parallel.*

Let the straight line  $EF$ , which falls upon the two straight lines  
 $AB$ ,  $CD$ , make the alternate angles  $AEF$ ,  $EFD$ , equal to one another.  
Then  $AB$  shall be parallel to  $CD$ .



For, if  $AB$  be not parallel to  $CD$ ;  
then  $AB$  and  $CD$  being produced will meet, either towards  $A$  and  $C$ ,  
or towards  $B$  and  $D$ .

Let  $AB$ ,  $CD$  be produced and meet, if possible, towards  $B$  and  $D$ ,  
in the point  $G$ ,

then  $GEF$  is a triangle.

And because a side  $GE$  of the triangle  $GEF$  is produced to  $A$ , therefore its exterior angle  $AEF$  is greater than the interior and opposite angle  $EFG$ ; (I. 16.)

but the angle  $AEF$  is equal to the angle  $EFG$ ; (hyp.)

therefore the angle  $AEF$  is greater than, and equal to, the angle  $EFG$ ; which is impossible.

Therefore  $AB$ ,  $CD$  being produced, do not meet towards  $B$ ,  $D$ .

In like manner, it may be demonstrated, that they do not meet when produced towards  $A$ ,  $C$ .

But those straight lines in the same plane, which meet neither way, though produced ever so far, are parallel to one another; (def. 35.)

therefore  $AB$  is parallel to  $CD$ .

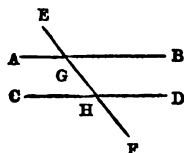
Wherefore, if a straight line, &c. Q.E.D.

### PROPOSITION XXVIII. THEOREM.

*If a straight line falling upon two other straight lines, make the exterior angle equal to the interior and opposite upon the same side of the line; or make the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.*

Let the straight line  $EF$ , which falls upon the two straight lines  $AB$ ,  $CD$ , make the exterior angle  $EGB$  equal to the interior and opposite angle  $GHD$ , upon the same side of the line  $EF$ ; or make the two interior angles  $BGH$ ,  $GHD$  on the same side together equal to two right angles.

Then  $AB$  shall be parallel to  $CD$ .



Because the angle  $EGB$  is equal to the angle  $GHD$ , (hyp.)

and the angle  $EGB$  is equal to the angle  $AGH$ , (I. 15.)

therefore the angle  $AGH$  is equal to the angle  $GHD$ ; (ax. 1.)

and they are alternate angles,

therefore  $AB$  is parallel to  $CD$ . (I. 27.)

Again, because the angles  $BGH$ ,  $GHD$  are together equal to two right angles, (hyp.)

and that the angles  $AGH$ ,  $BGH$  are also together equal to two right angles; (I. 13.)

therefore the angles  $AGH$ ,  $BGH$  are equal to the angles  $BGH$ ,  $GHD$ ; (ax. 1.)

take away from these equals, the common angle  $BGH$ ;

therefore the remaining angle  $AGH$  is equal to the remaining angle  $GHD$ ; (ax. 3.)

and they are alternate angles;

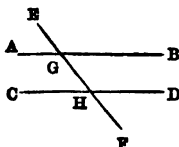
therefore  $AB$  is parallel to  $CD$ . (I. 27.)

Wherefore, if a straight line, &c. Q.E.D.

## PROPOSITION XXIX. THEOREM.

*If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.*

Let the straight line  $EF$  fall upon the parallel straight lines  $AB$ ,  $CD$ . Then the alternate angles  $AGH$ ,  $GHD$  shall be equal to one another; the exterior angle  $EGB$  shall be equal to the interior and opposite angle  $GHD$  upon the same side of the line  $EF$ ; and the two interior angles  $BGH$ ,  $GHD$  upon the same side of  $EF$  shall be together equal to two right angles.



First. For, if the angle  $AGH$  be not equal to the alternate angle  $GHD$ , one of them must be greater than the other;

if possible, let  $AGH$  be greater than  $GHD$ ,

then because the angle  $AGH$  is greater than the angle  $GHD$ ,

add to each of these unequals the angle  $BGH$ ;

therefore the angles  $AGH$ ,  $BGH$  are greater than the angles  $BGH$ ,  $GHD$ ; (ax. 4.)

but the angles  $AGH$ ,  $BGH$  are equal to two right angles; (i. 13.)

therefore the angles  $BGH$ ,  $GHD$  are less than two right angles;

but those straight lines, which with another straight line falling upon them, make the two interior angles on the same side less than two right angles, will meet together if continually produced; (ax. 12.)

therefore the straight lines  $AB$ ,  $CD$ , if produced far enough, will meet towards  $B$ ,  $D$ ;

but they never meet, since they are parallel by the hypothesis;

therefore the angle  $AGH$  is not unequal to the angle  $GHD$ ,

that is, the angle  $AGH$  is equal to the alternate angle  $GHD$ .

Secondly. Because the angle  $AGH$  is equal to the angle  $EGB$ , (i. 15.)

and the angle  $AGH$  is equal to the angle  $GHD$ ,

therefore the exterior angle  $EGB$  is equal to the interior and opposite angle  $GHD$ , on the same side of the line.

Thirdly. Because the angle  $EGB$  is equal to the angle  $GHD$ ,

add to each of them the angle  $BGH$ ;

therefore the angles  $EGB$ ,  $BGH$  are equal to the angles  $BGH$ ,  $GHD$ ; (ax. 2.)

but  $EGB$ ,  $BGH$  are equal to two right angles; (i. 13.)

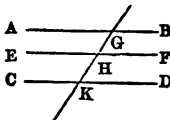
therefore also the two interior angles  $BGH$ ,  $GHD$  on the same side of the line are equal to two right angles. (ax. 1.)

Wherefore, if a straight line, &c. Q.E.D.

## PROPOSITION XXX. THEOREM.

*Straight lines which are parallel to the same straight line are parallel to each other.*

Let the straight lines  $AB$ ,  $CD$ , be each of them parallel to  $EF$ .  
Then shall  $AB$  be also parallel to  $CD$ .



Let the straight line  $GHK$  cut  $AB$ ,  $EF$ ,  $CD$ .

Then because  $GHK$  cuts the parallel straight lines  $AB$ ,  $EF$ , in  $G$ ,  $H$ :

therefore the angle  $AGH$  is equal to the alternate angle  $GHE$ . (I. 29.)

Again, because  $GHK$  cuts the parallel straight lines  $EF$ ,  $CD$ , in  $H$ ,  $K$ ;

therefore the exterior angle  $GHE$  is equal to the interior angle  $HKE$ ;

and it was shewn that the angle  $AGH$  is equal to the angle  $GHE$ ;

therefore the angle  $AGH$  is equal to the angle  $HKE$ ;

and these are alternate angles;

therefore  $AB$  is parallel to  $CD$ . (I. 27.)

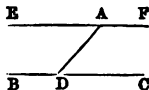
Wherefore, straight lines which are parallel, &c. Q. E. D.

## PROPOSITION XXXI. PROBLEM.

*To draw a straight line through a given point parallel to a given straight line.*

Let  $A$  be the given point, and  $BC$  the given straight line.

It is required to draw, through the point  $A$ , a straight line parallel to the straight line  $BC$ .



In the line  $BC$  take any point  $D$ , and join  $AD$ ;

at the point  $A$  in the straight line  $AD$ ,

make the angle  $DAE$  equal to the angle  $ADC$ , (I. 23.) on the opposite side of  $AD$ ;

and produce the straight line  $EA$  to  $F$ .

Then  $EF$  shall be parallel to  $BC$ .

Because the straight line  $AD$  meets the two straight lines  $EF$ ,  $BC$ , and makes the alternate angles  $EAD$ ,  $ADC$ , equal to one another,

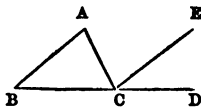
therefore  $EF$  is parallel to  $BC$ . (I. 27.)

Wherefore, through the given point  $A$ , has been drawn a straight line  $EAF$  parallel to the given straight line  $BC$ . Q. E. F.

## PROPOSITION XXXII. THEOREM.

*If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.*

Let  $ABC$  be a triangle, and let one of its sides  $BC$  be produced to  $D$ .  
Then the exterior angle  $ACD$  shall be equal to the two interior and opposite angles  $CAB, ABC$ :  
and the three interior angles  $ABC, BCA, CAB$  shall be equal to two right angles.



Through the point  $C$  draw  $CE$  parallel to the side  $BA$ . (I. 31.)

Then because  $CE$  is parallel to  $BA$ , and  $AC$  meets them, therefore the angle  $ACE$  is equal to the alternate angle  $BAC$ . (I. 29.)  
Again, because  $CE$  is parallel to  $AB$ , and  $BD$  falls upon them, therefore the exterior angle  $ECD$  is equal to the interior and opposite angle  $ABC$ ; (I. 29.)

but the angle  $ACE$  was shewn to be equal to the angle  $BAC$ ;  
therefore the whole exterior angle  $ACD$  is equal to the two interior and opposite angles  $CAB, ABC$ . (ax. 2.)

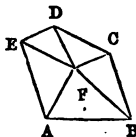
Again, because the angle  $ACD$  is equal to the two angles  $ABC, BAC$ ,  
to each of these equals add the angle  $ACB$ ,

therefore the angles  $ACD$  and  $ACB$  are equal to the three angles  $ABC, BAC$ , and  $ACB$ . (ax. 2.)

but the angles  $ACD, ACB$  are equal to two right angles, (I. 13.)  
therefore also the angles  $ABC, BAC, ACB$  are equal to two right angles. (ax. 1.)

Wherefore, if a side of any triangle be produced, &c. Q.E.D.

COR. 1. All the interior angles of any rectilineal figure together with four right angles, are equal to twice as many right angles as the figure has sides.



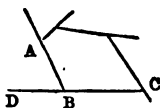
For any rectilineal figure  $ABCDE$  can be divided into as many triangles as the figure has sides, by drawing straight lines from a point  $F$  within the figure to each of its angles.

Then, because the three interior angles of a triangle are equal to two right angles, and there are as many triangles as the figure has sides, therefore all the angles of these triangles are equal to twice as many right angles as the figure has sides;

but the same angles of these triangles are equal to the interior angles of the figure together with the angles at the point  $F$ :

and the angles at the point  $F$ , which is the common vertex of all the triangles, are equal to four right angles, (I. 15. Cor. 2.) therefore the same angles of these triangles are equal to the angles of the figure together with four right angles; but it has been proved that the angles of the triangles are equal to twice as many right angles as the figure has sides; therefore all the angles of the figure together with four right angles, are equal to twice as many right angles as the figure has sides.

COR. 2. All the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles.



Since every interior angle  $ABC$  with its adjacent exterior angle  $ABD$ , is equal to two right angles, (I. 13.)

therefore all the interior angles, together with all the exterior angles, are equal to twice as many right angles as the figure has sides;

but it has been proved by the foregoing corollary, that all the interior angles together with four right angles are equal to twice as many right angles as the figure has sides;

therefore all the interior angles together with all the exterior angles, are equal to all the interior angles and four right angles, (ax. 1.)

take from these equals all the interior angles,

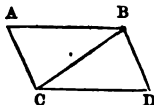
therefore all the exterior angles of the figure are equal to four right angles. (ax. 3.)

### PROPOSITION XXXIII. THEOREM.

*The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.*

Let  $AB$ ,  $CD$  be equal and parallel straight lines, and joined towards the same parts by the straight lines  $AC$ ,  $BD$ .

Then  $AC$ ,  $BD$  shall be equal and parallel.



Join  $BC$ .

Then because  $AB$  is parallel to  $CD$ , and  $BC$  meets them, therefore the angle  $ABC$  is equal to the alternate angle  $BCD$ ; (I. 29.)

and because  $AB$  is equal to  $CD$ , and  $BC$  common to the two triangles  $ABC$ ,  $DCB$ ; the two sides  $AB$ ,  $BC$ , are equal to the two  $DC$ ,  $CB$ , each to each, and the angle  $ABC$  was proved to be equal to the angle  $BCD$ :

therefore the base  $AC$  is equal to the base  $BD$ , (I. 4.)

and the triangle  $ABC$  to the triangle  $BCD$ ,

and the other angles to the other angles, each to each, to which the equal sides are opposite;

therefore the angle  $ACB$  is equal to the angle  $CBD$ .

And because the straight line  $BC$  meets the two straight lines  $AC$ ,  $BD$ , and makes the alternate angles  $ACB$ ,  $CBD$  equal to one another;

therefore  $AC$  is parallel to  $BD$ ; (I. 27.)

and  $AC$  was shewn to be equal to  $BD$ .

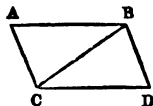
Therefore, straight lines which, &c. Q.E.D.

### PROPOSITION XXXIV. THEOREM.

*The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects it, that is, divides it into two equal parts.*

Let  $ACDB$  be a parallelogram, of which  $BC$  is a diameter.

Then the opposite sides and angles of the figure shall be equal to one another; and the diameter  $BC$  shall bisect it.



Because  $AB$  is parallel to  $CD$ , and  $BC$  meets them, therefore the angle  $ABC$  is equal to the alternate angle  $BCD$ . (I. 29.)

And because  $AC$  is parallel to  $BD$ , and  $BC$  meets them, therefore the angle  $ACB$  is equal to the alternate angle  $CBD$ . (I. 29.)

Hence in the two triangles  $ABC$ ,  $CBD$ , because the two angles  $ABC$ ,  $BCA$  in the one, are equal to the two angles  $BCD$ ,  $CBD$  in the other, each to each; and one side  $BC$ , which is adjacent to their equal angles, common to the two triangles;

therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other, (I. 26.)

namely, the side  $AB$  to the side  $CD$ , and  $AC$  to  $BD$ , and the angle  $BAC$  to the angle  $BDC$ .

And because the angle  $ABC$  is equal to the angle  $BCD$ ,

and the angle  $CBD$  to the angle  $ACB$ ,

therefore the whole angle  $ABD$  is equal to the whole angle  $ACD$ ; (ax. 2.)

and the angle  $BAC$  has been shewn to be equal to  $BDC$ ; therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diameter  $BC$  bisects it.

For since  $AB$  is equal to  $CD$ , and  $BC$  common, the two sides  $AB$ ,  $BC$ , are equal to the two  $DC$ ,  $CB$ , each to each,

and the angle  $ABC$  has been proved to be equal to the angle  $BCD$ ; therefore the triangle  $ABC$  is equal to the triangle  $BCD$ ; (I. 4.) and the diameter  $BC$  divides the parallelogram  $ACDB$  into two equal parts.

Q.E.D.

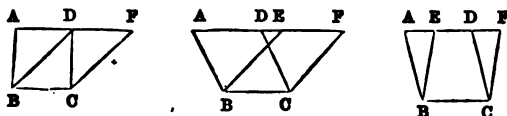


## PROPOSITION XXXV. THEOREM.

*Parallelograms upon the same base, and between the same parallels, are equal to one another.*

Let the parallelograms  $ABCD$ ,  $EBCF$  be upon the same base  $BC$ , and between the same parallels  $AF$ ,  $BC$ .

Then the parallelogram  $ABCD$  shall be equal to the parallelogram  $EBCF$ .



If the sides  $AD$ ,  $DF$  of the parallelograms  $ABCD$ ,  $EBCF$ , opposite to the base  $BC$ , be terminated in the same point  $D$ ; then it is plain that each of the parallelograms is double of the triangle  $BDC$ ; (I. 34.)

and therefore the parallelogram  $ABCD$  is equal to the parallelogram  $EBCF$ . (ax. 6.)

But if the sides  $AD$ ,  $EF$ , opposite to the base  $BC$ , be not terminated in the same point;

Then, because  $ABCD$  is a parallelogram,

therefore  $AD$  is equal to  $BC$ ; (I. 34.)

and for a similar reason,  $EF$  is equal to  $BC$ ;

wherefore  $AD$  is equal to  $EF$ ; (ax. 1.)

and  $DE$  is common;

therefore the whole, or the remainder  $AE$ , is equal to the whole, or the remainder  $DF$ ; (ax. 2 or 3.)

and  $AB$  is equal to  $DC$ ; (I. 24.)

hence in the triangles  $EAB$ ,  $FDC$ ,

because  $FD$  is equal to  $EA$ , and  $DC$  to  $AB$ ,

and the exterior angle  $FDC$  is equal to the interior and opposite angle  $EAB$ ; (I. 29.)

therefore the base  $FC$  is equal to the base  $EB$ ; (I. 4.)

and the triangle  $FDC$  is equal to the triangle  $EAB$ .

From the trapezium  $ABCF$  take the triangle  $FDC$ ,

and from the same trapezium take the triangle  $EAB$ ,

and the remainders are equal, (ax. 3.)

therefore the parallelogram  $ABCD$  is equal to the parallelogram  $EBCF$ .

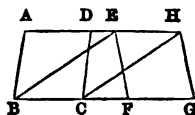
Therefore, parallelograms upon the same, &c. Q.E.D.

## PROPOSITION XXXVI. THEOREM.

*Parallelograms upon equal bases and between the same parallels, are equal to one another.*

Let  $ABCD$ ,  $EFGH$  be parallelograms upon equal bases  $BC$ ,  $FG$ , and between the same parallels  $AH$ ,  $BG$ .

Then the parallelogram  $ABCD$  shall be equal to the parallelogram  $EFGH$ .



Join  $BE$ ,  $CH$ .

Then because  $BC$  is equal to  $FG$ , (hyp.) and  $FG$  to  $EH$ , (I. 34.)

therefore  $BC$  is equal to  $EH$ ; (ax. 1.)

and these lines are parallels, and joined towards the same parts by the straight lines  $BE$ ,  $CH$ ;

but straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel; (I. 33.)

therefore  $BE$ ,  $CH$  are both equal and parallel;

wherefore  $EBCH$  is a parallelogram. (def. A.)

And because the parallelograms  $ABCD$ ,  $EBCH$ , are upon the same base  $BC$ , and between the same parallels  $BC$ ,  $AH$ ;

therefore the parallelogram  $ABCD$  is equal to the parallelogram  $EBCH$ . (I. 35.)

For the same reason, the parallelogram  $EFGH$  is equal to the parallelogram  $EBCH$ ;

therefore the parallelogram  $ABCD$  is equal to the parallelogram  $EFGH$ . (ax. 1.)

Therefore, parallelograms upon equal, &c. Q. E. D.

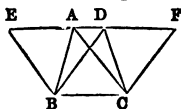
#### PROPOSITION XXXVII. THEOREM.

*Triangles upon the same base and between the same parallels, are equal to one another.*

Let the triangles  $ABC$ ,  $DBC$  be upon the same base  $BC$ ,

and between the same parallels  $AD$ ,  $BC$ .

Then the triangle  $ABC$  shall be equal to the triangle  $DBC$ .



Produce  $AD$  both ways to the points  $E$ ,  $F$ ;

through  $B$  draw  $BE$  parallel to  $CA$ , (I. 31.)

and through  $C$  draw  $CF$  parallel to  $BD$ .

Then each of the figures  $EBCA$ ,  $DBC F$  is a parallelogram; and  $EBCA$  is equal to  $DBC F$ , (I. 35.) because they are upon the same base  $BC$ , and between the same parallels  $BC$ ,  $EF$ .

And because the diameter  $AB$  bisects the parallelogram  $EBCA$ , therefore the triangle  $ABC$  is half of the parallelogram  $EBCA$ ; (I. 34.)

also because the diameter  $DC$  bisects the parallelogram  $DBC F$ ,

therefore the triangle  $DBC$  is half of the parallelogram  $DBC F$ ,

but the halves of equal things are equal; (ax. 7.)

therefore the triangle  $ABC$  is equal to the triangle  $DBC$ .

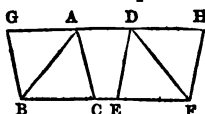
Wherefore, triangles, &c. Q. E. D.

## PROPOSITION XXXVIII. THEOREM.

*Triangles upon equal bases and between the same parallels, are equal to one another.*

Let the triangles  $ABC$ ,  $DEF$  be upon equal bases  $BC$ ,  $EF$ , and between the same parallels  $BF$ ,  $AD$ .

Then the triangle  $ABC$  shall be equal to the triangle  $DEF$ .



Produce  $AD$  both ways to the points  $G$ ,  $H$ ;  
through  $B$  draw  $BG$  parallel to  $CA$ , (I. 31.)  
and through  $F$  draw  $FH$  parallel to  $ED$ .

Then each of the figures  $GBCA$ ,  $DEFH$  is a parallelogram;  
and they are equal to one another, (I. 36.)  
because they are upon equal bases  $BC$ ,  $EF$ ,  
and between the same parallels  $BF$ ,  $GH$ .

And because the diameter  $AB$  bisects the parallelogram  $GBCA$ ,  
therefore the triangle  $ABC$  is the half of the parallelogram  $GBCA$ ;  
(I. 34.)

also, because the diameter  $DF$  bisects the parallelogram  $DEFH$ ,  
therefore the triangle  $DEF$  is the half of the parallelogram  $DEFH$ ;  
but the halves of equal things are equal; (ax. 7.)

therefore the triangle  $ABC$  is equal to the triangle  $DEF$ .

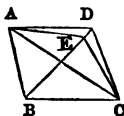
Wherefore, triangles upon equal bases, &c. Q.E.D.

## PROPOSITION XXXIX. THEOREM.

*Equal triangles upon the same base and upon the same side of it, are between the same parallels.*

Let the equal triangles  $ABC$ ,  $DBC$  be upon the same base  $BC$ ,  
and upon the same side of it.

Then the triangles  $ABC$ ,  $DBC$  shall be between the same parallels.



Join  $AD$ ; then  $AD$  shall be parallel to  $BC$ .

For if  $AD$  be not parallel to  $BC$ ,

if possible, through the point  $A$ , draw  $AE$  parallel to  $BC$ , (I. 31.)  
meeting  $BD$ , or  $BD$  produced, in  $E$ , and join  $EC$ .

Then the triangle  $ABC$  is equal to the triangle  $EBC$ , (I. 37.)

because they are upon the same base  $BC$ ,

and between the same parallels  $BC$ ,  $AE$ :

but the triangle  $ABC$  is equal to the triangle  $DBC$ ; (hyp.)

therefore the triangle  $DBC$  is equal to the triangle  $EBC$ ,

the greater triangle equal to the less, which is impossible:  
therefore  $AE$  is not parallel to  $BC$ .

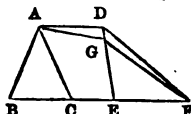
In the same manner it can be demonstrated,  
that no other line drawn from  $A$  but  $AD$  is parallel to  $BC$ ;  
 $AD$  is therefore parallel to  $BC$ .

Wherefore, equal triangles upon, &c. Q.E.D.

### PROPOSITION XL. THEOREM.

*Equal triangles upon equal bases in the same straight line, and towards the same parts, are between the same parallels.*

Let the equal triangles  $ABC$ ,  $DEF$  be upon equal bases  $BC$ ,  $EF$ , in the same straight line  $BF$ , and towards the same parts.  
Then they shall be between the same parallels.



Join  $AD$ ; then  $AD$  shall be parallel to  $BF$ .

For if  $AD$  be not parallel to  $BF$ ,

if possible, through  $A$  draw  $AG$  parallel to  $BF$ , (I. 31.)  
meeting  $ED$ , or  $ED$  produced in  $G$ , and join  $GF$ .

Then the triangle  $ABC$  is equal to the triangle  $GEF$ , (I. 38.)

because they are upon equal bases  $BC$ ,  $EF$ ,  
and between the same parallels  $BF$ ,  $AG$ ;

but the triangle  $ABC$  is equal to the triangle  $DEF$ ; (hyp.)  
therefore the triangle  $DEF$  is equal to the triangle  $GEF$ , (ax. 1.)  
the greater triangle equal to the less, which is impossible:  
therefore  $AG$  is not parallel to  $BF$ .

And in the same manner it can be demonstrated,  
that there is no other line drawn from  $A$  parallel to it but  $AD$ ;  
 $AD$  is therefore parallel to  $BF$ .

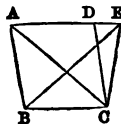
Wherefore, equal triangles upon, &c. Q.E.D.

### PROPOSITION XLI. THEOREM.

*If a parallelogram and a triangle be upon the same base, and between the same parallels; the parallelogram shall be double of the triangle.*

Let the parallelogram  $ABCD$ , and the triangle  $EBC$  be upon the same base  $BC$ , and between the same parallels  $BC$ ,  $AE$ .

Then the parallelogram  $ABCD$  shall be double of the triangle  $EBC$ .



Join  $AC$ .

Then the triangle  $ABC$  is equal to the triangle  $EBC$ , (I. 37.)

because they are upon the same base  $BC$ , and between the same parallels  $BC, AE$ .

But the parallelogram  $ABCD$  is double of the triangle  $ABC$ , because the diameter  $AC$  bisects it; (I. 34.)

wherefore  $ABCD$  is also double of the triangle  $EBC$ .

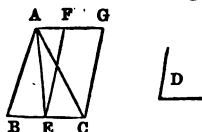
Therefore, if a parallelogram and a triangle, &c. Q.E.D.

### PROPOSITION XLII. PROBLEM.

*To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

Let  $ABC$  be the given triangle, and  $D$  the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the given triangle  $ABC$ , and have one of its angles equal to  $D$ .



Bisect  $BC$  in  $E$ , (I. 10.) and join  $AE$ ;

at the point  $E$  in the straight line  $EC$ ,

make the angle  $CEF$  equal to the angle  $D$ ; (I. 23.)

through  $C$  draw  $CG$  parallel to  $EF$ , and through  $A$  draw  $AFG$  parallel to  $BC$ , (I. 31.) meeting  $EF$  in  $F$ , and  $CG$  in  $G$ .

Then the figure  $CEFG$  is a parallelogram. (def. A.)

And because the triangles  $ABE, AEC$  are on the equal bases  $BE, EC$ , and between the same parallels  $BC, AG$ ;

they are therefore equal to one another; (I. 38.)

and the triangle  $ABC$  is double of the triangle  $AEC$ ;

but the parallelogram  $FECG$  is double of the triangle  $AEC$ , (I. 41.)

because they are upon the same base  $EC$ , and between the same parallels  $EC, AG$ ;

therefore the parallelogram  $FECG$  is equal to the triangle  $ABC$ , (ax. 6.)

and it has one of its angles  $CEF$  equal to the given angle  $D$ .

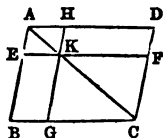
Wherefore, a parallelogram  $FECG$  has been described equal to the given triangle  $ABC$ , and having one of its angles  $CEF$  equal to the given angle  $D$ . Q.E.F.

### PROPOSITION XLIII. THEOREM.

*The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.*

Let  $ABCD$  be a parallelogram, of which the diameter is  $AC$ : and  $EH, GF$  the parallelograms about  $AC$ , that is, through which  $AC$  passes: also  $BK, KD$  the other parallelograms which make up the whole figure  $ABCD$ , which are therefore called the complements.

Then the complement  $BK$  shall be equal to the complement  $KD$ .



Because  $ABCD$  is a parallelogram, and  $AC$  its diameter, therefore the triangle  $ABC$  is equal to the triangle  $ADC$ . (I. 34.)

Again, because  $EKHA$  is a parallelogram, and  $AK$  its diameter, therefore the triangle  $AEK$  is equal to the triangle  $AHK$ ; (I. 34.) and for the same reason, the triangle  $KGC$  is equal to the triangle  $KFC$ .

Wherefore the two triangles  $AEK$ ,  $KGC$  are equal to the two triangles  $AHK$ ,  $KFC$ , (ax. 2.)

but the whole triangle  $ABC$  is equal to the whole triangle  $ADC$ ; therefore the remaining complement  $BK$  is equal to the remaining complement  $KD$ . (ax. 3.)

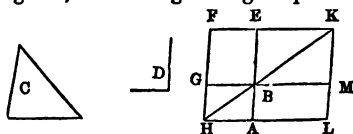
Wherefore the complements, &c. Q.E.D.

PROPOSITION XLIV. PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let  $AB$  be the given straight line, and  $C$  the given triangle, and  $D$  the given rectilineal angle.

It is required to apply to the straight line  $AB$ , a parallelogram equal to the triangle  $C$ , and having an angle equal to the angle  $D$ .



Make the parallelogram  $BEFG$  equal to the triangle  $C$ , and having the angle  $EBG$  equal to the angle  $D$ , (I. 42.)

so that  $BE$  be in the same straight line with  $AB$ ;

produce  $FG$  to  $H$ ,

through  $A$  draw  $AH$  parallel to  $BG$  or  $EF$ , (I. 31.) and join  $HB$ .

Then because the straight line  $HF$  falls upon the parallels  $AH$ ,  $EF$ , therefore the angles  $AHF$ ,  $HFE$  are together equal to two right angles; (I. 29.)

wherefore the angles  $BHF$ ,  $HFE$  are less than two right angles:

but straight lines which with another straight line, make the two interior angles upon the same side less than two right angles, do meet if produced far enough: (ax. 12.)

therefore  $HB$ ,  $FE$  shall meet if produced;

let them be produced and meet in  $K$ ,

through  $K$  draw  $KL$  parallel to  $EA$  or  $FH$ ,

and produce  $HA$ ,  $GB$  to meet  $KL$  in the points  $L$ ,  $M$ .

Then  $HLKF$  is a parallelogram, of which the diameter is  $HK$ ;

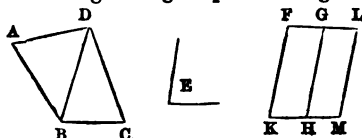
and  $AG$ ,  $ME$ , are the parallelograms about  $HK$ ;  
 also  $LB$ ,  $BF$  are the complements;  
 therefore the complement  $LB$  is equal to the complement  $BF$ ; (I. 43.)  
 but the complement  $BF$  is equal to the triangle  $C$ ; (constr.)  
 wherefore  $LB$  is equal to the triangle  $C$ .  
 And because the angle  $GBE$  is equal to the angle  $ABM$ , (I. 15.)  
 and likewise to the angle  $D$ ; (constr.)  
 therefore the angle  $ABM$  is equal to the angle  $D$ . (ax. 1.)  
 Therefore to the given straight line  $AB$ , the parallelogram  $LB$  has  
 been applied, equal to the triangle  $C$ , and having the angle  $ABM$   
 equal to the given angle  $D$ . Q. E. F.

### PROPOSITION XLV. PROBLEM.

*To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.*

Let  $ABCD$  be the given rectilineal figure, and  $E$  the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the figure  $ABCD$ , and having an angle equal to the given angle  $E$ .



Join  $DB$ .

Describe the parallelogram  $FH$  equal to the triangle  $ADB$ , and having the angle  $FKH$  equal to the angle  $E$ ; (I. 42.)

to the straight line  $GH$ , apply the parallelogram  $GM$  equal to the triangle  $DBC$ , having the angle  $GHM$  equal to the angle  $E$ . (I. 44.)

Then the figure  $FKML$  shall be the parallelogram required.

Because each of the angles  $FKH$ ,  $GHM$ , is equal to the angle  $E$ ,

therefore the angle  $FKH$  is equal to the angle  $GHM$ ;

add to each of these equals the angle  $KHG$ ;

therefore the angles  $FKH$ ,  $KHG$  are equal to the angles  $KHG$ ,  $GHM$ ;  
 but  $FKH$ ,  $KHG$  are equal to two right angles; (I. 29.)

therefore also  $KHG$ ,  $GHM$  are equal to two right angles;

and because at the point  $H$ , in the straight line  $GH$ , the two straight lines  $KH$ ,  $HM$ , upon the opposite sides of it, make the adjacent angles  $KHG$ ,  $GHM$  equal to two right angles,

therefore  $HK$  is in the same straight line with  $HM$ . (I. 14.)

And because the line  $HG$  meets the parallels  $KM$ ,  $FG$ ,

therefore the angle  $MHG$  is equal to the alternate angle  $HGF$ ; (I. 29.)

add to each of these equals the angle  $HGL$ ;

therefore the angles  $MHG$ ,  $HGL$  are equal to the angles  $HGF$ ,  $HGL$ ;  
 but the angles  $MHG$ ,  $HGL$  are equal to two right angles; (I. 29.)

therefore also the angles  $HGF$ ,  $HGL$  are equal to two right angles,  
 and therefore  $FG$  is in the same straight line with  $GL$ . (I. 14.)

And because  $KF$  is parallel to  $HG$ , and  $HG$  to  $ML$ ,  
 therefore  $KF$  is parallel to  $ML$ ; (I. 30.)  
 and  $FL$  has been proved parallel to  $KM$ ,  
 wherefore the figure  $FKML$  is a parallelogram;  
 and since the parallelogram  $HF$  is equal to the triangle  $ABD$ ,  
 and the parallelogram  $GM$  to the triangle  $BDC$ ;  
 therefore the whole parallelogram  $KFLM$  is equal to the whole  
 rectilineal figure  $ABCD$ .

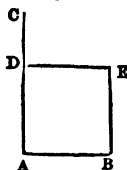
Therefore the parallelogram  $KFLM$  has been described equal to  
 the given rectilineal figure  $ABCD$ , having the angle  $FKM$  equal to  
 the given angle  $E$ . Q. E. F.

COR. From this it is manifest how, to a given straight line, to apply  
 a parallelogram, which shall have an angle equal to a given rectilineal  
 angle, and shall be equal to a given rectilineal figure; viz. by applying  
 to the given straight line a parallelogram equal to the first triangle  
 $ABD$ , (I. 44.) and having an angle equal to the given angle.

#### PROPOSITION XLVI. PROBLEM.

*To describe a square upon a given straight line.*

Let  $AB$  be the given straight line.



It is required to describe a square upon  $AB$ .

From the point  $A$  draw  $AC$  at right angles to  $AB$ ; (I. 11.)

make  $AD$  equal to  $AB$ ; (I. 3.)

through the point  $D$  draw  $DE$  parallel to  $AB$ ; (I. 31.)

and through  $B$ , draw  $BE$  parallel to  $AD$ , meeting  $DE$  in  $E$ ;

therefore  $ABED$  is a parallelogram;

whence  $AB$  is equal to  $DE$ , and  $AD$  to  $BE$ ; (I. 34.)

but  $AD$  is equal to  $AB$ ,

therefore the four lines  $AB$ ,  $BE$ ,  $ED$ ,  $DA$  are equal to one another,

and the parallelogram  $ABED$  is equilateral.

It has likewise all its angles right angles;

since  $AD$  meets the parallels  $AB$ ,  $DE$ ,

therefore the angles  $BAD$ ,  $ADE$  are equal to two right angles; (I. 29.)

but  $BAD$  is a right angle; (constr.)

therefore also  $ADE$  is a right angle.

But the opposite angles of parallelograms are equal; (I. 34.)

therefore each of the opposite angles  $ABE$ ,  $BED$  is a right angle;

wherefore the figure  $ABED$  is rectangular,

and it has been proved to be equilateral;

therefore the figure  $ABED$  is a square, (def. 30.)

and it is described upon the given straight line  $AB$ . Q. E. F.

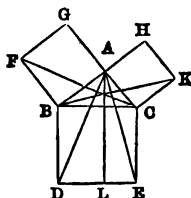


**COR.** Hence, every parallelogram that has one of its angles a right angle, has all its angles right angles.

**PROPOSITION XLVII. THEOREM.**

*In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.*

Let  $ABC$  be a right-angled triangle, having the right angle  $BAC$ .  
Then the square described upon the side  $BC$ , shall be equal to the squares described upon  $BA$ ,  $AC$ .



On  $BC$  describe the square  $BDEC$ , (I. 46.)

and on  $BA$ ,  $AC$  the squares  $GB$ ,  $HC$ ;

through  $A$  draw  $AL$  parallel to  $BD$  or  $CE$ ; (I. 31.)

and join  $AD$ ,  $FC$ .

Then because the angle  $BAC$  is a right angle, (hyp.)

and that the angle  $BAG$  is a right angle, (def. 30.)

the two straight lines  $AC$ ,  $AG$  upon the opposite sides of  $AB$ , make with it at the point  $A$ , the adjacent angles equal to two right angles; therefore  $CA$  is in the same straight line with  $AG$ . (I. 14.)

For the same reason,  $BA$  and  $AH$  are in the same straight line.

And because the angle  $DBC$  is equal to the angle  $FBA$ , each of them being a right angle,

add to each of these equals the angle  $ABC$ ,

therefore the whole angle  $ABD$  is equal to the whole angle  $FBC$ . (ax. 2.)

And because the two sides  $AB$ ,  $BD$ , are equal to the two sides  $FB$ ,  $BC$ , each to each, and the included angle  $ABD$  is equal to the included angle  $FBC$ ,

therefore the base  $AD$  is equal to the base  $FC$ , (I. 4.)

and the triangle  $ABD$  to the triangle  $FBC$ .

Now the parallelogram  $BL$  is double of the triangle  $ABD$ , (I. 41.)

because they are upon the same base  $BD$ , and between the same parallels  $BD$ ,  $AL$ ;

also the square  $GB$  is double of the triangle  $FBC$ ,

because these also are upon the same base  $FB$ , and between the same parallels  $FB$ ,  $GC$ .

But the doubles of equals are equal to one another; (ax. 6.)

therefore the parallelogram  $BL$  is equal to the square  $GB$ .

Similarly, by joining  $AE$ ,  $BK$ , it can be proved, that the parallelogram  $CL$  is equal to the square  $HC$ .

Therefore the whole square  $BDEC$  is equal to the two squares  $GB$ ,  $HC$ ; (ax. 2.)

and the square  $BDEC$  is described upon the straight line  $BC$ ,

and the squares  $GB$ ,  $HC$ , upon  $AB$ ,  $AC$ :

therefore the square upon the side  $BC$ , is equal to the squares upon the sides  $AB$ ,  $AC$ .

Therefore, in any right-angled triangle, &c. Q.E.D.

### PROPOSITION XLVIII. THEOREM.

*If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.*

Let the square described upon  $BC$ , one of the sides of the triangle  $ABC$ , be equal to the squares upon the other two sides,  $AB$ ,  $AC$ .

Then the angle  $BAC$  shall be a right angle.



From the point  $A$  draw  $AD$  at right angles to  $AC$ , (I. 11.)

make  $AD$  equal to  $AB$ , and join  $DC$ .

Then, because  $AD$  is equal to  $AB$ ,

the square on  $AD$  is equal to the square on  $AB$ ;

to each of these equals add the square on  $AC$ ;

therefore the squares on  $AD$ ,  $AC$  are equal to the squares on  $AB$ ,  $AC$ ;

but the squares on  $AD$ ,  $AC$  are equal to the square on  $DC$ , (I. 47.)

because the angle  $DAC$  is a right angle;

and the square on  $BC$ , by hypothesis, is equal to the squares on  $BA$ ,  $AC$ ;

therefore the square on  $DC$  is equal to the square on  $BC$ ;

and therefore the side  $DC$  is equal to the side  $BC$ .

And because the side  $AD$  is equal to the side  $AB$ ,

and  $AC$  is common to the two triangles  $DAC$ ,  $BAC$ ;

the two sides  $DA$ ,  $AC$ , are equal to the two  $BA$ ,  $AC$ , each to each;

and the base  $DC$  has been proved to be equal to the base  $BC$ ;

therefore the angle  $DAC$  is equal to the angle  $BAC$ ; (I. 8.)

but  $DAC$  is a right angle;

therefore also  $BAC$  is a right angle.

Therefore, if the square described upon, &c. Q.E.D.

## NOTES TO BOOK I.

### ON THE DEFINITIONS.

GEOMETRY is one of the most perfect of the deductive Sciences, and seems to rest on the simplest inductions from experience and observation.

The first principles of Geometry are therefore in this view consistent hypotheses founded on facts cognizable by the senses, and it is a subject of primary importance to draw a distinction between the conception of things and the things themselves. These hypotheses do not involve any property contrary to the real nature of the things, and consequently cannot be regarded as arbitrary, but in certain respects, agree with the conceptions which the things themselves suggest to the mind through the medium of the senses. The essential definitions of Geometry therefore being inductions from observation and experience, rest ultimately on the evidence of the senses.

It is by experience we become acquainted with the existence of individual forms of magnitudes; but by the mental process of abstraction, which begins with a particular instance, and proceeds to the general idea of all objects of the same kind, we attain to the general conception of those forms which come under the same general idea.

The essential definitions of Geometry express generalized conceptions of real existences in their most perfect ideal forms: the laws and appearances of nature, and the operations of the human intellect being supposed uniform and consistent.

But in cases where the subject falls under the class of simple ideas, the terms of the definitions so called, are no more than merely equivalent expressions. The simple idea described by a proper term or terms, does not in fact admit of definition properly so called. The definitions in Euclid's Elements may be divided into two classes, those which merely explain the meaning of the terms employed, and those, which, besides explaining the meaning of the terms, suppose the existence of the things described in the definitions.

Definitions in Geometry cannot be of such a form as to explain the nature and properties of the figures defined: it is sufficient that they give marks whereby the thing defined may be distinguished from every other of the same kind. It will at once be obvious, that the definitions of Geometry, one of the pure sciences, being abstractions of space, are not like the definitions in any one of the physical sciences. The discovery of any new physical facts may render necessary some alteration or modification in the definitions of the latter.

Def. 1. Simson has adopted Theon's definition of a point. Euclid's definition is, σημείον ἐστὶν οὐ μέρος οὐδέν, "A point is that, of which there is no part," or which cannot be parted or divided, as it is explained by Proclus. The Greek term σημείον, literally means, a visible sign or mark on a surface, in other words, a physical point. The English term point, means the sharp end of any thing, or a mark made by it. The word point comes from the Latin punctum, through the French word point. Neither of these terms, in its literal sense, appears to give a very exact notion of what is to be understood by a point in Geometry. Euclid's definition of a point merely expresses a negative property, which excludes the proper and literal meaning of the Greek term, as applied to denote a physical point, or a mark which is visible to the senses.

Pythagoras defined a point to be μονὰς θέσιν ἔχουσα, "a monad having position." By uniting the positive idea of position, with the negative idea of defect of magnitude, the conception of a point in Geometry may

be rendered perhaps more intelligible. A point is defined to be that which has no magnitude, but position only.

Def. II. Every visible line has both length and breadth, and it is impossible to draw any line whatever which shall have no breadth. The definition requires the conception of the length only of the line to be considered, abstracted from, and independently of, all idea of its breadth.

Def. III. This definition renders more intelligible the exact meaning of the definition of a point: and we may add, that, in the Elements, Euclid supposes that the intersection of two lines is a point, and that two lines can intersect each other in one point only.

Def. IV. The straight line or right line is a term so clear and intelligible as to be incapable of becoming more so by formal definition. Euclid's definition is *Εὐθεία γραμμὴ ἴστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται*, wherein he states it to lie *evenly*, or *equally*, or *upon an equality* (*ἐξ ἴσου*) between its extremities, and which Proclus explains as being stretched between its extremities, *ἢ ἢ ἀκρῶν τεταμένην*.

If the line be conceived to be drawn on a plane surface, the words *ἐξ ἴσου* may mean, that no part of the line which is called a straight line deviates either from one side or the other of the direction which is fixed by the extremities of the line; and thus it may be distinguished from a curved line, which does not lie, in this sense, evenly between its extreme points. If the line be conceived to be drawn in space, the words *ἐξ ἴσου*, must be understood to apply to every direction on every side of the line between its extremities.

Every straight line situated in a plane, is considered to have two sides; and when the direction of a line is known, the line is said to be given in position; also, when the length is known or can be found, it is said to be given in magnitude.

From the definition of a straight line, it follows, that two points fix a straight line in position, which is the foundation of the first and second postulates. Hence straight lines which are proved to coincide in two more points, are called, "one and the same straight line," Prop. 14, Book I, or, which is the same thing, that "two straight lines cannot have a common segment," as Simson shews in his Corollary to Prop. 11, Book I.

The following definition of straight lines has also been proposed. "Straight lines are those which, if they coincide in any two points, coincide as far as they are produced." But this is rather a criterion of straight lines, and analogous to the eleventh axiom, which states that, "all right angles are equal to one another," and suggests that all straight lines may be made to coincide wholly, if the lines be equal; or partially, if the lines be of unequal lengths. A definition should properly be restricted to the description of the thing defined, as it exists, independently of any comparison of its properties or of tacitly assuming the existence of axioms.

Def. VII. Euclid's definition of a plane surface is *Ἐπίπεδος ἐπιφάνεια ἴστιν ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται*, "A plane surface is that which lies evenly or equally with the straight lines in it;" instead of which Simson has given the definition which was originally proposed by Hero the Elder. A plane superficies may be supposed to be situated in any position, and to be continued in every direction to any extent.

Def. VIII. Simson remarks that this definition seems to include the angles formed by two curved lines, or a curve and a straight line, as well as that formed by two straight lines.

Angles made by straight lines only, are treated of in Elementary Geometry.

Def. ix. It is of the highest importance to attain a clear conception of an angle, and of the sum and difference of two angles. The literal meaning of the term *angulus* suggests the Geometrical conception of an angle, which may be regarded as formed by the divergence of two straight lines from a point. In the definition of an angle, the magnitude of the angle is independent of the lengths of the two lines by which it is included; their mutual divergence from the point at which they meet, is the criterion of the magnitude of an angle, as it is pointed out in the succeeding definitions. The point at which the two lines meet is called the angular point or the vertex of the angle, and must not be confounded with the magnitude of the angle itself. The right angle is fixed in magnitude, and, on this account, it is made the standard with which all other angles are compared.

Two straight lines which actually intersect one another, or which when produced would intersect, are said to be inclined to one another, and the inclination of the two lines is determined by the angle which they make with one another.

Def. x. It may be here observed that in the Elements, Euclid always assumes that when one line is perpendicular to another line, the latter is also perpendicular to the former; and always calls a *right angle*, ὀρθὴ γωνία; but a *straight line*, εὐθεῖα γραμμή.

Def. xix. This has been restored from Proclus, as it seems to have a meaning in the construction of Prop. 14, Book II; the first case of Prop. 33, Book III, and Prop. 13, Book VI. The definition of the segment of a circle is not once alluded to in Book I, and is not required before the discussion of the properties of the circle in Book III. Proclus remarks on this definition: "Hence you may collect that the center has three places: for it is either within the figure, as in the circle; or in its perimeter, as in the semicircle; or without the figure, as in certain conic lines."

Def. xxiv-xxix. Triangles are divided into three classes, by reference to the relations of their sides; and into three other classes, by reference to their angles. A further classification may be made by considering both the relation of the sides and angles in each triangle.

In Simson's definition of the isosceles triangle, the word *only* must be omitted, as in the Cor. Prop. 5, Book I, an isosceles triangle may be equilateral, and an equilateral triangle is considered isosceles in Prop. 15, Book IV. Objection has been made to the definition of an acute-angled triangle. It is said that it cannot be admitted as a definition, that all the three angles of a triangle are acute, which is supposed in Def. 29. It may be replied, that the definitions of the three kinds of angles point out and seem to supply a foundation for a similar distinction of triangles.

Def. xxx-xxxiv. The definitions of quadrilateral figures are liable to objection. All of them, except the trapezium, fall under the general idea of a parallelogram; but as Euclid defined parallel straight lines after he had defined four-sided figures, no other arrangement could be adopted than the one he has followed; and for which there appeared to him, without doubt, some probable reasons. Sir Henry Savile, in his Seventh Lecture, remarks on some of the definitions of Euclid, "Nec dissimulandum aliquot harum in manibus exiguum esse usum in Geometria." A few verbal emendations have been made in some of them.

A square is a four-sided plane figure having all its sides equal, and one angle a right angle: because it is proved in Prop. 46, Book I, that if a parallelogram have one angle a right angle, all its angles are right angles.

An oblong, in the same manner, may be defined as a plane figure of four sides, having only its opposite sides equal, and one of its angles a right angle.

A rhomboid is a four-sided plane figure having only its opposite sides equal to one another and its angles not right angles.

Sometimes an irregular four-sided figure which has two sides parallel, is called a trapezoid.

Def. xxxv. It is possible for two right lines never to meet when produced, and not be parallel.

Def. A. The term parallelogram literally implies a figure formed by parallel straight lines, and may consist of four, six, eight, or any even number of sides, where every two of the opposite sides are parallel to one another. In the *Elements*, however, the term is restricted to four-sided figures, and includes the four species of figures named in the Definitions xxx—xxxiii.

The synthetic method is followed by Euclid not only in the demonstrations of the propositions, but also in laying down the definitions. He commences with the simplest abstractions, defining a point, a line, an angle, a superficies, and their different varieties. This mode of proceeding involves the difficulty, almost insurmountable, of defining satisfactorily the elementary abstractions of Geometry. It has been observed, that it is necessary to consider a solid, that is, a magnitude which has length, breadth, and thickness, in order to understand aright the definitions of a point, a line, and a superficies. A solid or volume considered apart from its physical properties, suggests the idea of the surfaces by which it is bounded: a surface, the idea of the line or lines which form its boundaries: and a finite line, the points which form its extremities. A solid is therefore bounded by surfaces; a surface is bounded by lines; and a line is terminated by two points. A point marks position only: a line has one dimension, length only, and defines distance: a superficies has two dimensions, length and breadth, and defines extension: and a solid has three dimensions, length, breadth, and thickness, and defines some portion of space.

It may also be remarked that two points are sufficient to determine the position of a straight line, and three points not in the same straight line, are necessary to fix the position of a plane.

### ON THE POSTULATES.

THE definitions assume the possible existence of straight lines and circles, and the postulates predicate the possibility of drawing and of producing straight lines, and of describing circles. The postulates form the principles of construction assumed in the *Elements*; and are, in fact, problems, the possibility of which is admitted to be self-evident, and to require no proof.

It must, however, be carefully remarked, that the third postulate only admits that when any line is given in position and magnitude, a circle may be described from either extremity of the line as a center, and with a radius equal to the length of the line, as in *Euc. i. 1*. It does not admit the description of a circle with any other point as a center than one of the extremities of the given line.

*Euc. i. 2*, shews how, from any given point, to draw a straight line equal to another straight line which is given in magnitude and position.

## ON THE AXIOMS.

AXIOMS are usually defined to be self-evident truths, which cannot be rendered more evident by demonstration; in other words, the axioms of Geometry are theorems, the truth of which is admitted without proof. It is by experience we first become acquainted with the different forms of geometrical magnitudes, and the axioms, or the fundamental ideas of their equality or inequality appear to rest on the same basis. The conception of the truth of the axioms does not appear to be more removed from experience than the conception of the definitions.

These axioms, or first principles of demonstration, are such theorems as cannot be resolved into simpler theorems, and no theorem ought to be admitted as a first principle of reasoning which is capable of being demonstrated. An axiom, and (when it is convertible) its converse, should both be of such a nature as that neither of them should require a formal demonstration.

The first and most simple idea, derived from experience is, that every magnitude fills a certain space, and that several magnitudes may successively fill the same space.

All the knowledge we have of magnitude is purely relative, and the most simple relations are those of equality and inequality. In the comparison of magnitudes, some are considered as given or known, and the unknown are compared with the known, and conclusions are synthetically deduced with respect to the equality or inequality of the magnitudes under consideration. In this manner we form our idea of equality, which is thus formally stated in the eighth axiom: "Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another."

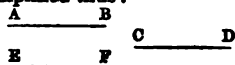
Every specific definition is referred to this universal principle. With regard to a few more general definitions which do not furnish an equality, it will be found that some hypothesis is always made reducing them to that principle, before any theory is built upon them. As for example, the definition of a straight line is to be referred to the tenth axiom; the definition of a right angle to the eleventh axiom; and the definition of parallel straight lines to the twelfth axiom.

The eighth axiom is called the principle of superposition, or, the mental process by which one Geometrical magnitude may be conceived to be placed on another, so as exactly to coincide with it, in the parts which are made the subject of comparison. Thus, if one straight line be conceived to be placed upon another, so that their extremities are coincident, the two straight lines are equal. If the directions of two lines which include one angle, coincide with the directions of the two lines which contain another angle, where the points, from which the angles diverge, coincide, then the two angles are equal: the lengths of the lines not affecting in any way the magnitudes of the angles. When one plane figure is conceived to be placed upon another, so that the boundaries of one exactly coincide with the boundaries of the other, then the two plane figures are equal. It may also be remarked, that the converse of this proposition is not universally true, namely, that when two magnitudes are equal, they coincide with one another: since two magnitudes may be equal in area, as two parallelograms or two triangles, Euc. I. 35, 37; but their boundaries may not be equal: and, consequently, by superposition, the figures could not exactly coincide: all such figures, however, having equal areas, by a different arrangement of their parts, may be made to coincide exactly.

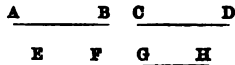
This axiom is the criterion of Geometrical equality, and is essentially different from the criterion of Arithmetical equality. Two geometrical magnitudes are equal, when they coincide or may be made to coincide: two abstract numbers are equal, when they contain the same aggregate of units; and two concrete numbers are equal, when they contain the same number of units of the same kind of magnitude. It is at once obvious, that Arithmetical representations of Geometrical magnitudes are not admissible in Euclid's criterion of Geometrical Equality, as he has not fixed the unit of magnitude of either the straight line, the angle, or the superficies. Perhaps Euclid intended that the first seven axioms should be applicable to numbers as well as to Geometrical magnitudes, and this is in accordance with the words of Proclus, who calls the axioms, *common notions*, not peculiar to the subject of Geometry.

Several of the axioms may be generally exemplified thus:

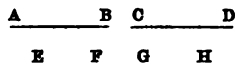
Axiom I. If the straight line  $AB$  be equal to the straight line  $CD$ ; and if the straight line  $EF$  be also equal to the straight line  $CD$ ; then the straight line  $AB$  is equal to the straight line  $EF$ .



Axiom II. If the line  $AB$  be equal to the line  $CD$ ; and if the line  $EF$  be also equal to the line  $GH$ ; then the sum of the lines  $AB$  and  $EF$  is equal to the sum of the lines  $CD$  and  $GH$ .

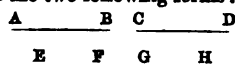


Axiom III. If the line  $AB$  be equal to the line  $CD$ ; and if the line  $EF$  be also equal to the line  $GH$ ; then the difference of  $AB$  and  $EF$ , is equal to the difference of  $CD$  and  $GH$ .

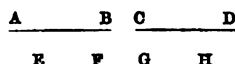


Axiom IV. admits of being exemplified under the two following forms:

1. If the line  $AB$  be equal to the line  $CD$ ; and if the line  $EF$  be greater than the line  $GH$ ; then the sum of the lines  $AB$  and  $EF$  is greater than the sum of the lines  $CD$  and  $GH$ .

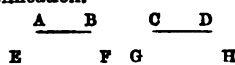


2. If the line  $AB$  be equal to the line  $CD$ ; and if the line  $EF$  be less than the line  $GH$ ; then the sum of the lines  $AB$  and  $EF$  is less than the sum of the lines  $CD$  and  $GH$ .

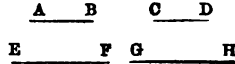


Axiom V. also admits of two forms of exemplification.

1. If the line  $AB$  be equal to the line  $CD$ ; and if the line  $EF$  be greater than the line  $GH$ ; then the difference of the lines  $AB$  and  $EF$  is greater than the difference of  $CD$  and  $GH$ .

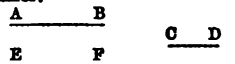


2. If the line  $AB$  be equal to the line  $CD$ ; and if the line  $EF$  be less than the line  $GH$ ; then the difference of the lines  $AB$  and  $EF$  is less than the difference of the lines  $CD$  and  $GH$ .



The axiom, "If unequals be taken from equals, the remainders are unequal," may be exemplified in the same manner.

Axiom VI. If the line  $AB$  be double of the line  $CD$ ; and if the line  $EF$  be also double of the line  $CD$ ;



then the line  $AB$  is equal to the line  $EF$ .

Axiom VII. If the line  $AB$  be the half of the line  $CD$ ; and if the line  $EF$  be also the half of the line  $CD$ ;



then the line  $AB$  is equal to the line  $EF$ .



It may be observed that when equal magnitudes are taken from unequal magnitudes, the greater remainder exceeds the less remainder by as much as the greater of the unequal magnitudes exceeds the less.

If unequals be taken from unequals, the remainders are not always unequal; they may be equal: also if unequals be added to unequals the wholes are not always unequal, they may also be equal.

**Axiom ix.** The whole is greater than its part, and conversely, the part is less than the whole. This axiom appears to assert the contrary of the eighth axiom, namely, that two magnitudes, of which one is greater than the other, cannot be made to coincide with one another.

**Axiom x.** The property of straight lines expressed by the tenth axiom, namely, "that two straight lines cannot enclose a space," is obviously implied in the definition of straight lines; for if they enclosed a space, they could not coincide between their extreme points, when the two lines are equal.

**Axiom xi.** This axiom has been asserted to be a demonstrable theorem. As an angle is a species of magnitude, this axiom is only a particular application of the eighth axiom to right angles.

**Axiom xii.** See the notes on Prop. xxix. Book i.

### ON THE PROPOSITIONS.

WHENEVER a judgment is formally expressed, there must be something respecting which the judgment is expressed, and something else which constitutes the judgment. The former is called the *subject* of the proposition, and the latter, the *predicate*, which may be anything which can be affirmed or denied respecting the *subject*.

The propositions in Euclid's Elements of Geometry may be divided into two classes, *problems* and *theorems*. A proposition, as the term imports, is something proposed; it is a *problem*, when some Geometrical construction is required to be effected: and it is a *theorem* when some Geometrical property is to be demonstrated. Every proposition is naturally divided into two parts; a problem consists of the *data*, or things given; and the *quæsitæ*, or things required: a theorem, consists of the *subject* or *hypothesis*, and the *conclusion*, or *predicate*. Hence the distinction between a problem and a theorem is this, that a problem consists of the data and the quæsitæ, and requires solution: and a theorem consists of the hypothesis and the predicate, and requires demonstration.

All propositions are *affirmative* or *negative*; that is, they either assert some property, as Euc. i. 4, or deny the existence of some property, as Euc. i. 7; and every proposition which is affirmatively stated has a contradictory corresponding proposition. If the affirmative be proved to be true, the contradictory is false.

All propositions may be viewed as (1) *universally affirmative*, or *universally negative*; (2) as *particularly affirmative*, or *particularly negative*.

The connected course of reasoning by which any Geometrical truth is established is called a demonstration. It is called a *direct* demonstration when the predicate of the proposition is inferred directly from the premises, as the conclusion of a series of successive deductions. The demonstration is called *indirect*, when the conclusion shows that the introduction of any other supposition contrary to the hypothesis stated in the proposition, necessarily leads to an absurdity.

It has been remarked by Pascal, that "Geometry is almost the only subject as to which we find truths wherein all men agree: and one cause of this is, that Geometers alone regard the true laws of demonstration."

These are enumerated by him as eight in number. "1. To define nothing which cannot be expressed in clearer terms than those in which it is already expressed. 2. To leave no obscure or equivocal terms undefined. 3. To employ in the definition no terms not already known. 4. To omit nothing in the principles from which we argue, unless we are sure it is granted. 5. To lay down no axiom which is not perfectly evident. 6. To demonstrate nothing which is as clear already as we can make it. 7. To prove every thing in the least doubtful by means of self-evident axioms, or of propositions already demonstrated. 8. To substitute mentally the definition instead of the thing defined." Of these rules, he says, "the first, fourth and sixth are not absolutely necessary to avoid error, but the other five are indispensable; and though they may be found in books of logic, none but the Geometers have paid any regard to them."

The course pursued in the demonstrations of the propositions in Euclid's Elements of Geometry, is always to refer directly to some expressed principle, to leave nothing to be inferred from vague expressions, and to make every step of the demonstrations the object of the understanding.

It has been maintained by some philosophers, that a genuine definition contains some property or properties which can form a basis for demonstration, and that the science of Geometry is deduced from the definitions, and that on them alone the demonstrations depend. Others have maintained that a definition explains only the meaning of a term, and does not embrace the nature and properties of the thing defined.

If the propositions usually called postulates and axioms are either tacitly assumed or expressly stated in the definitions; in this view, demonstrations may be said to be legitimately founded on definitions. If, on the other hand, a definition is simply an explanation of the meaning of a term, whether abstract or concrete, by such marks as may prevent a misconception of the thing defined; it will be at once obvious that some constructive and theoretic principles must be assumed, besides the definitions to form the ground of legitimate demonstration. These principles we conceive to be the postulates and axioms. The postulates describe constructions which may be admitted as possible by direct appeal to our experience; and the axioms assert general theoretic truths so simple and self-evident as to require no proof, but to be admitted as the assumed first principles of demonstration. Under this view all Geometrical reasonings proceed upon the admission of the hypotheses assumed in the definitions, and the unquestioned possibility of the postulates, and the truth of the axioms.

Deductive reasoning is generally delivered in the form of an enthymeme, or an argument wherein one enunciation is not expressed, but is readily supplied by the reader: and it may be observed, that although this is the ordinary mode of speaking and writing, it is not in the strictly syllogistic form; as either the *major* or the *minor* premiss only is formally stated before the conclusion: Thus in Euc. I. 1.

Because the point *A* is the center of the circle *BCD*;

therefore the straight line *AB* is equal to the straight line *AC*.

The premiss here omitted, is: all straight lines drawn from the center of a circle to the circumference are equal.

In a similar way may be supplied the reserved premiss in every enthymeme. The conclusion of two enthymemes may form the major and minor premiss of a third syllogism, and so on, and thus any process of reasoning is reduced to the strictly syllogistic form. And in this way it is shewn

that the general theorems of Geometry are demonstrated by means of syllogisms founded on the axioms and definitions.

Every syllogism consists of three propositions, of which, two are called the premisses, and the third, the conclusion. These propositions contain three terms, the subject and predicate of the conclusion, and the middle term which connects the predicate and the conclusion together. The subject of the conclusion is called *the minor*, and the predicate of the conclusion is called *the major term*, of the syllogism. The major term appears in one premiss, and the minor term in the other, with the middle term, which is in both premisses. That premiss which contains the middle term and the major term, is called *the major premiss*; and that which contains the middle term and the minor term, is called *the minor premiss* of the syllogism. As an example, we may take the syllogism in the demonstration of Prop. 1, Book 1, wherein it will be seen that the middle term is the subject of the major premiss and the predicate of the minor.

Major premiss: because the straight line  $AB$  is equal to the straight line  $AC$ ;  
Minor premiss: and, because the straight line  $BC$  is equal to the straight line  $AB$ ;

Conclusion: therefore the straight line  $BC$  is equal to the straight line  $AC$ .

Here,  $BC$  is the subject, and  $AC$  the predicate of the conclusion.

$BC$  is the subject, and  $AB$  the predicate of the minor premiss.

$AB$  is the subject, and  $AC$  the predicate of the major premiss.

Also,  $AC$  is the major term,  $BC$  the minor term, and  $AB$  the middle term of the syllogism.

In this syllogism, it may be remarked that the definition of a straight line is assumed, and the definition of the Geometrical equality of two straight lines; also that a general theoretic truth, or axiom, forms the ground of the conclusion. And further, though it be impossible to make any point, mark or sign (*σημείον*) which has not both length and breadth, and any line which has not both length and breadth; the demonstrations in Geometry do not on this account become invalid. For they are pursued on the hypothesis that the point has no parts, but position only: and the line has length only, but no breadth or thickness: also that the surface has length and breadth only, but no thickness: and all the conclusions at which we arrive are independent of every other consideration.

The truth of the conclusion in the syllogism depends upon the truth of the premisses. If the premisses, or only one of them be not true, the conclusion is false. The conclusion is said to *follow from* the premisses; whereas, in truth, it is *contained in* the premisses. The expression must be understood of the mind apprehending in succession, the truth of the premisses, and subsequent to that, the truth of the conclusion; so that the conclusion *follows from* the premisses in order of time as far as reference is made to the mind's apprehension of the whole argument.

Every proposition, when complete, may be divided into six parts, as Proclus has pointed out in his commentary.

1. *The proposition, or general enunciation*, which states in general terms the conditions of the problem or theorem.

2. *The exposition, or particular enunciation*, which exhibits the subject of the proposition in particular terms as a fact, and refers it to some diagram described.

3. *The determination* contains the *predicate* in particular terms, as it is pointed out in the diagram, and directs attention to the demonstration, by pronouncing the thing sought.

4. *The construction* applies the postulates to prepare the diagram for the demonstration.

5. *The demonstration* is the connexion of syllogisms, which prove the truth or falsehood of the theorem, the possibility or impossibility of the problem, in that particular case exhibited in the diagram.

6. *The conclusion* is merely the repetition of the general enunciation, wherein the predicate is asserted as a demonstrated truth.

Prop. i. In the first two Books, the circle is employed as a mechanical instrument, in the same manner as the straight line, and the use made of it rests entirely on the third postulate. No properties of the circle are discussed in these books beyond the definition and the third postulate. When two circles are described, one of which has its center in the circumference of the other, the two circles being each of them partly within and partly without the other, their circumferences must intersect each other in two points; and it is obvious from the two circles cutting each other, in two points, one on each side of the given line, that two equilateral triangles may be formed on the given line.

Prop. ii. When the given point is neither in the line, nor in the line produced, this problem admits of eight different lines being drawn from the given point in different directions, every one of which is a solution of the problem. For, 1. The given line has two extremities, to each of which a line may be drawn from the given point. 2. The equilateral triangle may be described on either side of this line. 3. And the side  $BD$  of the equilateral triangle  $ABD$  may be produced either way.

But when the given point lies either in the line or in the line produced, the distinction which arises from joining the two ends of the line with the given point, no longer exists, and there are only four cases of the problem.

The construction of this problem assumes a neater form, by first describing the circle  $CGH$  with center  $B$  and radius  $BC$ , and producing  $DB$  the side of the equilateral triangle  $DBA$  to meet the circumference in  $G$ : next, with center  $D$  and radius  $DG$ , describing the circle  $GKL$ , and then producing  $DA$  to meet the circumference in  $L$ .

By a similar construction the less of two given straight lines may be produced, so that the less together with the part produced may be equal to the greater.

Prop. iii. This problem admits of two solutions, and it is left undetermined from which end of the greater line the part is to be cut off.

By means of this problem, a straight line may be found equal to the sum or the difference of two given lines.

Prop. iv. This forms the first case of equal triangles, two other cases are proved in Prop. viii. and Prop. xxvi.

The term *base* is obviously taken from the idea of a building, and the same may be said of the term *altitude*. In Geometry, however, these terms are not restricted to one particular position of a figure, as in the case of a building, but may be in any position whatever.

Prop. v. Proclus has given, in his commentary, a proof for the equality of the angles at the base, without producing the equal sides. The construction follows the same order, taking in  $AB$  one side of the isosceles triangle  $ABC$ , a point  $D$  and cutting off from  $AC$  a part  $AE$  equal to  $AD$ , and then joining  $CD$  and  $BE$ .

A corollary is a theorem which results from the demonstration of a proposition.

Prop. vi. is the converse of one part of Prop. v. One proposition

is defined to be the *converse* of another when the hypothesis of the former becomes the predicate of the latter; and vice versa.

There is besides this, another kind of conversion, when a theorem has several hypotheses and one predicate; by assuming the predicate and one, or more than one of the hypotheses, some one of the hypotheses may be inferred as the predicate of the converse. In this manner, Prop. viii. is the converse of Prop. iv. It may here be observed, that converse theorems are not universally true: as for instance, the following direct proposition is universally true; "If two triangles have their three sides respectively equal, the three angles of each shall be respectively equal." But the converse is not universally true; namely, "If two triangles have the three angles in each respectively equal, the three sides are respectively equal." Converse theorems require, in some instances, the consideration of other conditions than those which enter into the proof of the direct theorem. *Converse* and *contrary* propositions are by no means to be confounded; the *contrary* proposition denies what is asserted, or asserts what is denied, in the *direct* proposition, but the subject and predicate in each are the same. A *contrary proposition* is a *completely contradictory proposition*, and the distinction consists in this—that *two contrary propositions* may both be false, but of *two contradictory propositions*, one of them must be true, and the other false. It may here be remarked, that one of the most common intellectual mistakes of learners, is to imagine that the denial of a proposition is a legitimate ground for affirming the contrary as true: whereas the rules of sound reasoning allow that the affirmation of a proposition as true, only affords a ground for the denial of the contrary as false.

Prop. vi. is the first instance of indirect demonstrations, and they are more suited for the proof of converse propositions. All those propositions which are demonstrated *ex absurdo*, are properly analytical demonstrations, according to the Greek notion of analysis, which first supposed the thing required, to be done, or to be true, and then shewed the consistency or inconsistency of this construction or hypothesis with truths admitted or already demonstrated.

In indirect demonstrations, where hypotheses are made which are not true and contrary to the truth stated in the proposition, it seems desirable that a form of expression should be employed different from that in which the hypotheses are true. In all cases therefore, whether noted by Euclid or not, the words *if possible* have been introduced; or some such qualifying expression, as in Euc. i. 6, so as not to leave upon the mind of the learner, the impression that the hypothesis which contradicts the proposition, is really true.

Prop. viii. When the three sides of one triangle are shewn to coincide with the three sides of any other, the equality of the triangles is at once obvious. This, however, is not stated at the conclusion of Prop. viii. or of Prop. xxvi. For the equality of the areas of two coincident triangles, reference is always made by Euclid to Prop. iv.

A direct demonstration may be given of this proposition, and Prop. vii. may be dispensed with altogether.

Let the triangles  $ABC$ ,  $DEF$  be so placed that the base  $BC$  may coincide with the base  $EF$ , and the vertices  $A$ ,  $D$  may be on opposite sides of  $EF$ . Join  $AD$ . Then because  $EAD$  is an isosceles triangle, the angle  $EAD$  is equal to the angle  $EDA$ ; and because  $CDA$  is an isosceles triangle, the angle  $CAD$  is equal to the angle  $CDA$ . Hence

the angle  $EAF$  is equal to the angle  $EDF$ , (ax. 2 or 3): or the angle  $BDC$  is equal to the angle  $EDF$ .

Prop. ix. If  $BA$ ,  $AC$  be in the same straight line. This problem then becomes the same as Prob. xi, which may be regarded as drawing a line which bisects an angle equal to two right angles.

If  $FA$  be produced in the fig. Prop. 9, it bisects the angle which is the defect of the angle  $BAC$  from four right angles.

By means of this problem, any angle may be divided into four, eight, sixteen, &c. equal angles.

Prop. x. A finite straight line may, by this problem, be divided into four, eight, sixteen, &c. equal parts.

Prop. xi. When the point is at the extremity of the line; by the second postulate the line may be produced, and then the construction applies. See note on Euc. iii. 31.

The distance between two points is the straight line which joins the points; but the distance between a point and a straight line, is the shortest line which can be drawn from the point to the line.

From this Prop. it follows that only one perpendicular can be drawn from a given point to a given line; and this perpendicular may be shewn to be less than any other line which can be drawn from the given point to the given line: and of the rest, the line which is nearer to the perpendicular is less than one more remote from it: also only two equal straight lines can be drawn from the same point to the line, one on each side of the perpendicular or the least. This property is analogous to Euc. iii. 7, 8.

The corollary to this proposition is not in the Greek text, but was added by Simson, who states that it "is necessary to Prop. 1, Book xi., and otherwise."

Prop. xii. The third postulate requires that the line  $CD$  should be drawn before the circle can be described with the center  $C$ , and radius  $CD$ .

Prop. xiv. is the converse of Prop. xiii. "Upon the opposite sides of it." If these words were omitted, it is possible for two lines to make with a third, two angles, which together are equal to two right angles, in such a manner that the two lines shall not be in the same straight line.

The line  $BE$  may be supposed to fall above, as in Euclid's figure, or below the line  $BD$ , and the demonstration is the same in form.

Prop. xv. is the development of the definition of an angle. If the lines at the angular point be produced, the produced lines have the same inclination to one another as the original lines, but in a different position.

The converse of this Proposition is not proved by Euclid, namely:—If the vertical angles made by four straight lines at the same point be respectively equal to each other, each pair of opposite lines shall be in the same straight line.

Prop. xvii. appears to be only a corollary to the preceding proposition, and it seems to be introduced to explain Axiom xii, of which it is the converse. The exact truth respecting the angles of a triangle is proved in Prop. xxxii.

Prop. xviii. It may here be remarked, for the purpose of guarding the student against a very common mistake, that in this proposition and in the converse of it, the *hypothesis* is stated before the *predicate*.

Prop. xix. is the converse of Prop. xviii. It may be remarked, that Prop. xix. bears the same relation to Prop. xviii., as Prop. vi. does to Prop. v.

Prop. xx. The following corollary arises from this proposition:—

A straight line is the shortest distance between two points. For the straight line  $BC$  is always less than  $BA$  and  $AC$ , however near the point  $A$  may be to the line  $BC$ .

It may be easily shewn from this proposition, that the difference of any two sides of a triangle is less than the third side.

Prop. xxii. When the sum of two of the lines is equal to, and when it is less than, the third line; let the diagrams be described, and they will exhibit the impossibility implied by the restriction laid down in the Proposition.

The same remark may be made here, as was made under the first Proposition, namely:—if one circle lies partly within and partly without another circle, the circumferences of the circles intersect each other in two points.

Prop. xxiii.  $CD$  might be taken equal to  $CE$ , and the construction effected by means of an isosceles triangle. It would, however, be less general than Euclid's, but is more convenient in practice.

Prop. xxiv. Simson makes the angle  $EDG$  at  $D$  in the line  $ED$ , the side which is not the greater of the two  $ED$ ,  $DF$ ; otherwise, three different cases would arise, as may be seen by forming the different figures. The point  $G$  might fall below or upon the base  $EF$  produced as well as above it. Prop. xxiv. and Prop. xxv. bear to each other the same relation as Prop. iv. and Prop. viii.

Prop. xxvi. This forms the third case of the equality of two triangles. Every triangle has three sides and three angles, and when any three of one triangle are given equal to any three of another, the triangles may be proved to be equal to one another, whenever the three magnitudes given in the hypothesis are independent of one another. Prop. iv. contains the first case, when the hypothesis consists of two sides and the included angle of each triangle. Prop. viii. contains the second, when the hypothesis consists of the three sides of each triangle. Prop. xxvi. contains the third, when the hypothesis consists of two angles, and one side either adjacent to the equal angles, or opposite to one of the equal angles in each triangle. There is another case, not proved by Euclid, when the hypothesis consists of two sides and one angle in each triangle, but these not the angles included by the two given sides in each triangle. This case however is only true under a certain restriction, thus:

*If two triangles have two sides of one of them equal to two sides of the other, each to each, and have also the angles opposite to one of the equal sides in each triangle, equal to one another, and if the angles opposite to the other equal sides be both acute, or both obtuse angles; then shall the third sides be equal in each triangle, as also the remaining angles of the one to the remaining angles of the other.*

Let  $ABC$ ,  $DEF$  be two triangles which have the sides  $AB$ ,  $AC$  equal to the two sides  $DE$ ,  $DF$ , each to each, and the angle  $ABC$  equal to the angle  $DEF$ : then, if the angles  $ACB$ ,  $DFE$ , be both acute, or both obtuse angles, the third side  $BC$  shall be equal to the third side  $EF$ , and also the angle  $BCA$  to the angle  $EFD$ , and the angle  $BAC$  to the angle  $EDF$ .

First. Let the angles  $ACB$ ,  $DFE$  opposite to the equal sides  $AB$ ,  $DE$ , be both acute angles.

If  $BC$  be not equal to  $EF$ , let  $BC$  be the greater, and from  $BC$ , cut off  $BG$  equal to  $EF$ , and join  $AG$ .

Then in the triangles  $ABG$ ,  $DEF$ , Euc. i. 4.  $AG$  is equal to  $DF$ ,

and the angle  $AGB$  to  $DFE$ . But since  $AC$  is equal to  $DF$ ,  $AG$  is equal to  $AC$ : and therefore the angle  $ACG$  is equal to the angle  $AGC$ , which is also an acute angle. But because  $AGC$ ,  $AGB$  are together equal to two right angles, and that  $AGC$  is an acute angle,  $AGB$  must be an obtuse angle; which is absurd. Wherefore,  $BC$  is not unequal to  $EF$ , that is,  $BC$  is equal to  $EF$ , and also the remaining angles of one triangle to the remaining angles of the other.

Secondly. Let the angles  $ACB$ ,  $DFE$ , be both *obtuse angles*. By proceeding in a similar way, it may be shewn that  $BC$  cannot be otherwise than equal to  $EF$ .

If  $ACB$ ,  $DFE$  be both *right angles*: the case falls under *Euc. i. 26*.

Prop. xxvii. Alternate angles are defined to be the two angles which two straight lines make with another at its extremities, but upon opposite sides of it.

When a straight line intersects two other straight lines, two pairs of alternate angles are formed by the lines at their intersections, as in the figure,  $BEF$ ,  $EFC$  are alternate angles as well as the angles  $AEF$ ,  $EFD$ .

Prop. xxviii. One angle is called "the exterior angle," and another "the interior and opposite angle," when they are formed on the same side of a straight line which falls upon or intersects two other straight lines. It is also obvious that on each side of the line, there will be two exterior and two interior and opposite angles. The exterior angle  $EGB$  has the angle  $GHD$  for its corresponding interior and opposite angle: also the exterior angle  $FHD$  has the angle  $HGB$  for its interior and opposite angle.

Prop. xxix is the converse of Prop. xxvii and Prop. xxviii.

As the definition of parallel straight lines simply describes them by a statement of the negative property, that they never meet; it is necessary that some positive property of parallel lines should be assumed as an axiom, on which reasonings on such lines may be founded.

Euclid has assumed the statement in the twelfth axiom, which has been objected to, as not being self-evident. A stronger objection appears to be, that the converse of it forms *Euc. i. 17*; for both the assumed axiom and its converse, should be so obvious as not to require formal demonstration.

Simson has attempted to overcome the objection, not by any improved definition and axiom respecting parallel lines; but, by considering Euclid's twelfth axiom to be a theorem, and for its proof, assuming two definitions and one axiom, and then demonstrating five subsidiary Propositions.

Instead of Euclid's twelfth axiom, the following has been proposed as a more simple property for the foundation of reasonings on parallel lines; namely, "If a straight line fall on two parallel straight lines, the alternate angles are equal to one another." In whatever this may exceed Euclid's definition in simplicity, it is liable to a similar objection, being the converse of *Euc. i. 27*.

Professor Playfair has adopted in his *Elements of Geometry*, that "Two straight lines which intersect one another cannot be both parallel to the same straight line." This apparently more simple axiom follows as a direct inference from *Euc. i. 30*.

But one of the least objectionable of all the definitions which have been proposed on this subject, appears to be that which simply expresses the conception of equidistance. It may be formally stated thus: "Parallel lines are such as lie in the same plane, and which neither recede from, nor approach to, each other." This includes the con-



ception stated by Euclid, that parallel lines never meet. Dr. Wallis observes on this subject, "*Parallelismus et æquidistantia vel idem sunt, vel certe se mutuo comitantur.*"

As an additional reason for this definition being preferred, it may be remarked that the meaning of the terms *γρῦμαὶ παράλληλοι*, suggests the exact idea of such lines.

An account of thirty methods which have been proposed at different times for avoiding the difficulty in the twelfth axiom, will be found in the appendix to Colonel Thompson's "*Geometry without Axioms.*"

Prop. xxx. In the diagram, the two lines  $AB$  and  $CD$  are placed one on each side of the line  $EF$ : the proposition may also be proved when both  $AB$  and  $CD$  are on the same side of  $EF$ .

Prop. xxxii. From this proposition, it is obvious that if one angle of a triangle be equal to the sum of the other two angles, that angle is a right angle, as is shewn in Euc. iii. 31, and that each of the angles of an equilateral triangle, is equal to two thirds of a right angle, as it is shewn in Euc. iv. 15. Also, if one angle of an isosceles triangle be a right angle, then each of the equal angles is half a right angle, as in Euc. ii. 9.

The three angles of a triangle may be shewn to be equal to two right angles without producing a side of the triangle, by drawing through any angle of the triangle a line parallel to the opposite side, as Proclus has remarked in his Commentary on this proposition. It is manifest from this proposition, that the third angle of a triangle is not independent of the sum of the other two; but is known if the sum of any two is known. Cor. 1 may be also proved by drawing lines from any one of the angles of the figure to the other angles. If any of the sides of the figure bend inwards and form what are called re-entering angles, the enunciation of these two corollaries will require some modification. As Euclid gives no definition of re-entering angles, it may fairly be concluded, he did not intend to enter into the proofs of the properties of figures which contain such angles.

Prop. xxxiii. The words "towards the same parts" are a necessary restriction: for if they were omitted, it would be doubtful whether the extremities  $A$ ,  $C$ , and  $B$ ,  $D$  were to be joined by the lines  $AC$  and  $BD$ ; or the extremities  $A$ ,  $D$ , and  $B$ ,  $C$ , by the lines  $AD$  and  $BC$ .

Prop. xxxiv. If the other diameter be drawn, it may be shewn that the diameters of a parallelogram bisect each other, as well as bisect the area of the parallelogram. If the parallelogram be right angled, the diagonals are equal; if the parallelogram be a square or a rhombus, the diagonals bisect each other at right angles. The converse of this Prop., namely, "If the opposite sides or opposite angles of a quadrilateral figure be equal, the opposite sides shall also be parallel; that is, the figure shall be a parallelogram," is not proved by Euclid.

Prop. xxxv. The latter part of the demonstration is not expressed very intelligibly. Simson, who altered the demonstration, seems in fact to consider two trapeziums of the same form and magnitude, and from one of them, to take the triangle  $ABE$ ; and from the other, the triangle  $DCF$ ; and then the remainders are equal by the third axiom: that is, the parallelogram  $ABCD$  is equal to the parallelogram  $EBCF$ . Otherwise, the triangle, whose base is  $DE$ , (fig. 2.) is taken twice from the trapezium, which would appear to be impossible, if the sense in which Euclid applies the third axiom, is to be retained here.

It may be observed, that the two parallelograms exhibited in fig. 2 partially lie on one another, and that the triangle whose base is  $BC$  is a common part of them, but that the triangle whose base is  $DE$  is entirely without both the parallelograms. After having proved the triangle  $ABE$  equal to the triangle  $DCF$ , if we take from these equals (fig. 2.) the triangle whose base is  $DE$ , and to each of the remainders add the triangle whose base is  $BC$ , then the parallelogram  $ABCD$  is equal to the parallelogram  $EBCF$ . In fig. 3, the equality of the parallelograms  $ABCD$ ,  $EBCF$ , is shewn by adding the figure  $EBCD$  to each of the triangles  $ABE$ ,  $DCF$ .

In this proposition, the word *equal* assumes a new meaning, and is no longer restricted to mean coincidence in all the parts of two figures.

Prop. xxxviii. In this proposition, it is to be understood that the bases of the two triangles are in the same straight line. If in the diagram the point  $E$  coincide with  $C$ , and  $D$  with  $A$ , then the angle of one triangle is supplemental to the other. Hence the following property:—If two triangles have two sides of the one respectively equal to two sides of the other, and the contained angles supplemental, the two triangles are equal.

A distinction ought to be made between *equal* triangles and *equivalent* triangles, the former including those whose sides and angles mutually coincide, the latter those whose areas only are equivalent.

Prop. xxxix. If the vertices of all the equal triangles which can be described upon the same base, or upon the equal bases as in Prop. 40, be joined, the line thus formed will be a straight line, and is called the locus of the vertices of equal triangles upon the same base, or upon equal bases.

A locus in plane Geometry is a straight line or a plane curve, every point of which and none else satisfies a certain condition. With the exception of the straight line and the circle, the two most simple loci; all other loci, perhaps including also the Conic Sections, may be more readily and effectually investigated algebraically by means of their rectangular or polar equations.

Prop. xli. The converse of this proposition is not proved by Euclid; viz. If a parallelogram is double of a triangle, and they have the same base, or equal bases upon the same straight line, and towards the same parts, they shall be between the same parallels. Also, it may easily be shewn that if two equal triangles are between the same parallels; they are either upon the same base, or upon equal bases.

Prop. xlii. A parallelogram described on a straight line is said to be *applied* to that line.

Prop. xlii. The problem is solved only for a rectilineal figure of four sides. If the given rectilineal figure have more than four sides, it may be divided into triangles by drawing straight lines from any angle of the figure to the opposite angles, and then a parallelogram equal to the third triangle can be applied to  $LM$ , and having an angle equal to  $E$ ; and so on for all the triangles of which the rectilineal figure is composed.

Prop. xlii. The square being considered as an equilateral rectangle, its area or surface may be expressed numerically if the number of lineal units in a side of the square be given, as is shewn in the note on Prop. I., Book II.

The student will not fail to remark the analogy which exists between the area of a square and the product of two equal numbers; and between the side of a square and the square root of a number. There is, however,

this distinction to be observed; it is always possible to find the product of two equal numbers, (or to find the square of a number, as it is usually called,) and to describe a square on a given line; but conversely, though the side of a given square is known from the figure itself, the exact number of units in the side of a square of given area, can only be found exactly, in such cases where the given number is a square number. For example, if the area of a square contain 9 square units, then the square root of 9 or 3, indicates the number of lineal units in the side of that square. Again, if the area of a square contain 12 square units, the side of the square is greater than 3, but less than 4 lineal units, and there is no number which will exactly express the side of that square: an approximation to the true length, however, may be obtained to any assigned degree of accuracy.

Prop. XLVII. In a right-angled triangle, the side opposite to the right angle is called the hypotenuse, and the other two sides, the base and perpendicular, according to their position.

In the diagram the three squares are described on the *outer* sides of the triangle  $ABC$ . The Proposition may also be demonstrated (1) when the three squares are described upon the *inner* sides of the triangle: (2) when one square is described on the outer side and the other two squares on the inner sides of the triangle: (3) when one square is described on the inner side and the other two squares on the outer sides of the triangle.

As one instance of the third case. If the square  $BE$  on the hypotenuse be described on the inner side of  $BC$  and the squares  $BG$ ,  $HC$  on the outer sides of  $AB$ ,  $AC$ ; the point  $D$  falls on the side  $FG$  (Euclid's fig.) of the square  $BG$ , and  $KH$  produced meets  $CE$  in  $E$ . Let  $LA$  meet  $BC$  in  $M$ . Join  $DA$ ; then the square  $GB$  and the oblong  $LB$  are each double of the triangle  $DAB$ , (Euc. I. 41.); and similarly by joining  $EA$ , the square  $HC$  and oblong  $LC$  are each double of the triangle  $EAC$ . Whence it follows that the squares on the sides  $AB$ ,  $AC$  are together equal to the square on the hypotenuse  $BC$ .

By this proposition may be found a square equal to the sum of any given squares, or equal to any multiple of a given square: or equal to the difference of two given squares.

The truth of this proposition may be exhibited to the eye in some particular instances. As in the case of that right-angled triangle whose three sides are 3, 4, and 5 units respectively. If through the points of division of two contiguous sides of each of the squares upon the sides, lines be drawn parallel to the sides (see the notes on Book II.), it will be obvious, that the squares will be divided into 9, 16 and 25 small squares, each of the same magnitude; and that the number of the small squares into which the squares on the perpendicular and base are divided is equal to the number into which the square on the hypotenuse is divided.

Prop. XLVIII is the converse of Prop. XLVII. In this Prop. is assumed the Corollary that "the squares described upon two equal lines are equal," and the converse, which properly ought to have been appended to Prop. XLVI.

The First Book of Euclid's Elements, it has been seen, is conversant with the construction and properties of rectilineal figures. It first lays down the definitions which limit the subjects of discussion in the First Book, next the three postulates, which restrict the instruments by which the constructions in Plane Geometry are effected; and thirdly, the twelve axioms, which express the principles by which a comparison is made between the ideas of the things defined.

This Book may be divided into three parts. The first part treats of the origin and properties of triangles, both with respect to their sides and angles; and the comparison of these mutually, both with regard to equality and inequality. The second part treats of the properties of parallel lines and of parallelograms. The third part exhibits the connection of the properties of triangles and parallelograms, and the equality of the squares on the base and perpendicular of a right-angled triangle to the square on the hypotenuse.

When the propositions of the First Book have been read with the notes, the student is recommended to use different letters in the diagrams, and where it is possible, diagrams of a form somewhat different from those exhibited in the text, for the purpose of testing the accuracy of his knowledge of the demonstrations. And further, when he has become sufficiently familiar with the method of geometrical reasoning, he may dispense with the aid of letters altogether, and acquire the power of expressing in general terms the process of reasoning in the demonstration of any proposition. Also, he is advised to answer the following questions before he attempts to apply the principles of the First Book to the solution of Problems and the demonstration of Theorems.

### QUESTIONS ON BOOK I.

1. What is the name of the Science of which Euclid gives the Elements? What is meant by *Solid Geometry*? Is there any distinction between *Plane Geometry*, and the *Geometry of Planes*?
2. Define the term *magnitude*, and specify the different kinds of magnitude considered in Geometry. What dimensions of space belong to figures treated of in the first six Books of Euclid?
3. Give Euclid's definition of a "straight line." What does he really use as his test of rectilinearity, and where does he first employ it? What objections have been made to it, and what substitute has been proposed as an available definition? How many points are necessary to fix the position of a straight line in a plane? When is one straight line said to *cut*, and when to *meet* another?
4. What positive property has a Geometrical point? From the definition of a straight line, shew that the intersection of two lines is a point.
5. Give Euclid's definition of a plane rectilineal angle. What are the limits of the angles considered in Geometry? Does Euclid consider angles greater than two right angles?
6. When is a straight line said to be drawn at *right angles*, and when *perpendicular*, to a given straight line?
7. Define a *triangle*; shew how many kinds of triangles there are according to the variation both of the *angles*, and of the *sides*.
8. What is Euclid's definition of a circle? Point out the assumption involved in your definition. Is any axiom implied in it? Shew that in this as in all other definitions some geometrical fact is assumed as somehow previously known.
9. Define the quadrilateral figures mentioned by Euclid.
10. Describe briefly the use and foundation of definitions, axioms, and postulates: give illustrations by an instance of each.
11. What objection may be made to the method and order in which Euclid has laid down the elementary abstractions of the Science of Geometry? What other method has been suggested?

12. What distinctions may be made between definitions in the Science of Geometry and in the Physical Sciences?

13. What is necessary to constitute an exact definition? Are definitions propositions? Are they arbitrary? Are they convertible? Does a Mathematical definition admit of proof on the principles of the Science to which it relates?

14. Enumerate the principles of construction assumed by Euclid.

15. Of what instruments may the use be considered to meet approximately the demands of Euclid's postulates? Why only *approximately*?

16. "A circle may be described from any center, with any straight line as radius." How does this postulate differ from Euclid's, and which of his problems is assumed in it?

17. What principles in the Physical Sciences correspond to axioms in Geometry?

18. Enumerate Euclid's twelve axioms and point out those which have special reference to Geometry. State the converse of those which admit of being so expressed.

19. What two tests of equality are assumed by Euclid? Is the assumption of the principle of superposition (ax. 8.), essential to all Geometrical reasoning? Is it correct to say, that it is "an appeal, though of the most familiar sort, to external observation"?

20. Could any, and if any, which of the axioms of Euclid be turned into definitions; and with what advantages or disadvantages?

21. Define the terms, Problem, Postulate, Axiom and Theorem. Are any of Euclid's axioms improperly so called?

22. Of what two parts does the enunciation of a Problem, and of a Theorem consist? Distinguish them in Euc. I. 4, 5, 18, 19.

23. When is a problem said to be indeterminate? Give an example.

24. When is one proposition said to be the converse or reciprocal of another? Give examples. Are converse propositions universally true? If not, under what circumstances are they necessarily true? Why is it necessary to demonstrate converse propositions? How are they proved?

25. Explain the meaning of the word *proposition*. Distinguish between *converse* and *contrary* propositions, and give examples.

26. State the grounds as to whether Geometrical reasonings depend for their conclusiveness upon axioms or definitions.

27. Explain the meaning of *enthymeme* and *syllogism*. How is the enthymeme made to assume the form of the syllogism? Give examples.

28. What constitutes a demonstration? State the laws of demonstration.

29. What are the principle parts, in the entire process of establishing a proposition?

30. Distinguish between a *direct* and *indirect* demonstration.

31. What is meant by the term *synthesis*, and what, by the term, *analysis*? Which of these modes of reasoning does Euclid adopt in his Elements of Geometry?

32. In what sense is it true that the conclusions of Geometry are necessary truths?

33. Enunciate those Geometrical definitions which are used in the proof of the propositions of the First Book.

34. If in Euclid I. 1, an equal triangle be described on the other side of the given line, what figure will the two triangles form?

35. In the diagram, Euclid I. 2, if  $DB$  a side of the equilateral triangle  $DAB$  be produced both ways and cut the circle whose center is  $B$  and radius  $BC$  in two points  $G$  and  $H$ ; shew that either of the dis-

tances  $DG, DH$  may be taken as the radius of the second circle; and give the proof in each case.

36. Explain how the propositions Euc. I. 2, 3, are rendered necessary by the restriction imposed by the third postulate. Is it necessary for the proof, that the triangle described in Euc. I. 2, should be equilateral? Could we, at this stage of the subject, describe an isosceles triangle on a given base?

37. State how Euc. I. 2, may be extended to the following problem: "From a given point to draw a straight line in a given direction equal to a given straight line."

38. How would you cut off from a straight line unlimited in both directions, a length equal to a given straight line?

39. In the proof of Euclid I. 4, how much depends upon Definition, how much upon Axiom?

40. Draw the figure for the third case of Euc. I. 7, and state why it needs no demonstration.

41. In the construction Euclid I. 9, is it indifferent in all cases on which side of the joining line the equilateral triangle is described?

42. Shew how a given straight line may be bisected by Euc. I. 1.

43. In what cases do the lines which bisect the interior angles of plane triangles, also bisect one, or more than one of the corresponding opposite sides of the triangles?

44. "Two straight lines cannot have a common segment." Has this corollary been tacitly assumed in any preceding proposition?

45. In Euc. I. 12, must the given line necessarily be "of unlimited length"?

46. Shew that (fig. Euc. I. 11) every point without the perpendicular drawn from the middle point of every straight line  $DE$ , is at unequal distances from the extremities  $D, E$  of that line.

47. From what proposition may it be inferred that a straight line is the shortest distance between two points?

48. Enunciate the propositions you employ in the proof of Euc. I. 16.

49. Is it essential to the truth of Euc. I. 21, that the two straight lines be drawn from the extremities of the base?

50. In the diagram, Euc. I. 21, by how much does the greater angle  $BDC$  exceed the less  $BAC$ ?

51. To form a triangle with three straight lines, any two of them must be greater than the third: is a similar limitation necessary with respect to the three angles?

52. Is it possible to form a triangle with three lines whose lengths are 1, 2, 3 units: or one with three lines whose lengths are 1,  $\sqrt{2}$ ,  $\sqrt{3}$ ?

53. Is it possible to construct a triangle whose angles shall be as the numbers 1, 2, 3? Prove or disprove your answer.

54. What is the reason of the limitation in the construction of Euc. I. 24, viz. "that  $DE$  is that side which is not greater than the other?"

55. Quote the first proposition in which the equality of two areas which cannot be superposed on each other is considered.

56. Is the following proposition universally true? "If two plane triangles have three elements of the one respectively equal to three elements of the other, the triangles are equal in every respect." Enumerate all the cases in which this equality is proved in the First Book. What case is omitted?

57. What parts of a triangle must be given in order that the triangle may be described?

58. State the converse of the second case of Euc. i. 26? Under what limitations is it true? Prove the proposition so limited?

59. Shew that the angle contained between the perpendiculars drawn to two given straight lines which meet each other, is equal to the angle contained by the lines themselves.

60. Are two triangles necessarily equal in all respects, where a side and two angles of the one are equal to a side and two angles of the other, each to each?

61. Illustrate fully the difference between analytical and synthetical proofs. What propositions in Euclid are demonstrated analytically?

62. Can it be properly predicated of any two straight lines that they never meet if indefinitely produced either way, antecedently to our knowledge of some other property of such lines, which makes the property first predicated of them a necessary conclusion from it?

63. Enunciate Euclid's definition and axiom relating to parallel straight lines; and state in what Props. of Book i. they are used.

64. What proposition is the converse to the twelfth axiom of the First Book? What other two propositions are complementary to these?

65. If lines being produced ever so far do not meet; can they be otherwise than parallel? If so, under what circumstances?

66. Define *adjacent angles*, *opposite angles*, *vertical angles*, and *alternate angles*; and give examples from the First Book of Euclid.

67. Can you suggest anything to justify the assumption in the twelfth axiom upon which the proof of Euc. i. 29, depends?

68. What objections have been urged against the definition and the doctrine of parallel straight lines as laid down by Euclid? Where does the difficulty originate? What other assumptions have been suggested and for what reasons?

69. Assuming as an axiom that two straight lines which cut one another cannot both be parallel to the same straight line; deduce Euclid's twelfth axiom as a corollary of Euc. i. 29.

70. From Euc. i. 27, shew that the distance between two parallel straight lines is constant?

71. If two straight lines be not parallel, shew that all straight lines falling on them, make alternate angles, which differ by the same angle.

72. Taking as the definition of parallel straight lines that they are equally inclined to the same straight line towards the same parts; prove that "being produced ever so far both ways they do not meet?" Prove also Euclid's axiom 12, by means of the same definition.

73. What is meant by *exterior* and *interior* angles? Point out examples.

74. Can the three angles of a triangle be proved equal to two right angles without producing a side of the triangle?

75. Shew how the corners of a triangular piece of paper may be turned down, so as to exhibit to the eye that the three angles of a triangle are equal to two right angles.

76. Explain the meaning of the term *corollary*. Enunciate the two corollaries appended to Euc. i. 32, and give another proof of the first. What other corollaries may be deduced from this proposition?

77. Shew that the two lines which bisect the exterior and interior angles of a triangle, as well as those which bisect any two interior angles of a parallelogram, contain a right angle.

78. The opposite sides and angles of a parallelogram are equal to one another, and the diameters bisect it. State and prove the converse of this proposition. Also shew that a quadrilateral figure, is a paral-

lelogram, when its diagonals bisect each other: and when its diagonals divide it into four triangles, which are equal, two and two, viz. those which have the same vertical angles.

79. If two straight lines join the extremities of two parallel straight lines, but *not* towards the same parts, when are the joining lines equal, and when are they unequal?

80. If either diameter of a four-sided figure divide it into two equal triangles, is the figure necessarily a parallelogram? Prove your answer.

81. Shew how to divide one of the parallelograms in *Euc. i. 35*, by straight lines so that the parts when properly arranged shall make up the other parallelogram.

82. Distinguish between *equal* triangles and *equivalent* triangles, and give examples from the First Book of Euclid.

83. What is meant by the locus of a point? Adduce instances of loci from the first Book of Euclid.

84. How is it shewn that equal triangles upon the same base or equal bases have equal altitudes, whether they are situated on the same or opposite sides of the same straight line?

85. In *Euc. i. 37, 38*, if the triangles are not towards the same parts, shew that the straight line joining the vertices of the triangles is bisected by the line containing the bases.

86. If the complements (*fig. Euc. i. 43*) be squares, determine their relation to the whole parallelogram.

87. What is meant by a parallelogram being applied to a straight line?

88. Is the proof of *Euc. i. 45*, perfectly general?

89. Define a square without including superfluous conditions, and explain the mode of constructing a square upon a given straight line in conformity with such a definition.

90. The sum of the angles of a square is equal to four right angles. Is the converse true? If not, why?

91. Conceiving a square to be a figure bounded by four equal straight lines not necessarily in the same plane, what condition respecting the angles is necessary to complete the definition?

92. In *Euclid i. 47*, why is it necessary to prove that one side of each square described upon each of the sides containing the right angle, should be in the same straight line with the other side of the triangle?

93. On what assumption is an analogy shewn to exist between the product of two equal numbers and the surface of a square?

94. Is the triangle whose sides are 3, 4, 5 right-angled, or not?

95. Can the side and diagonal of a square be represented simultaneously by any finite numbers?

96. By means of *Euc. i. 47*, the square roots of the natural numbers, 1, 2, 3, 4, &c. may be represented by straight lines.

97. If the square on the hypotenuse in the *fig. Euc. i. 47*, be described on the other side of it: shew from the diagram how the squares on the two sides of the triangle may be made to cover exactly the square on the hypotenuse.

98. If *Euclid ii. 2*, be assumed, enunciate the form in which *Euc. i. 47* may be expressed.

99. Classify all the properties of *triangles* and *parallelograms*, proved in the First Book of Euclid.

100. Mention any propositions in Book i. which are included in more general ones which follow.



## ON THE ANCIENT GEOMETRICAL ANALYSIS.

**SYNTHESIS**, or the method of composition, is a mode of reasoning which begins with something given, and ends with something required, either to be done or to be proved. This may be termed a *direct process*, as it leads from principles to consequences.

**Analysis**, or the method of resolution, is the reverse of synthesis, and thus it may be considered an *indirect process*, a method of reasoning from consequences to principles.

The synthetic method is pursued by Euclid in his Elements of Geometry. He commences with certain assumed principles, and proceeds to the solution of problems and the demonstration of theorems by undeniable and successive inferences from them.

The Geometrical Analysis was a process employed by the ancient Geometers, both for the discovery of the solution of problems and for the investigation of the truth of theorems. In the analysis of a *problem*, the *quæsitæ*, or what is required to be done, is supposed to have been effected, and the consequences are traced by a series of geometrical constructions and reasonings, till at length they terminate in the data of the problem, or in some previously demonstrated or admitted truth, whence the direct solution of the problem is deduced.

In the Synthesis of a *problem*, however, the last consequence of the analysis is assumed as the first step of the process, and by proceeding in a contrary order through the several steps of the analysis until the process terminate in the *quæsitæ*, the solution of the problem is effected.

But if, in the analysis, we arrive at a consequence which contradicts any truth demonstrated in the Elements, or which is inconsistent with the data of the problem, the problem must be impossible: and further, if in certain relations of the given magnitudes the construction be possible, while in other relations it is impossible, the discovery of these relations will become a necessary part of the solution of the problem.

In the analysis of a *theorem*, the question to be determined, is, whether by the application of the geometrical truths proved in the Elements, the predicate is consistent with the hypothesis. This point is ascertained by assuming the predicate to be true, and by deducing the successive consequences of this assumption combined with proved geometrical truths, till they terminate in the hypothesis of the theorem or some demonstrated truth. The theorem will be proved synthetically by retracing, in order, the steps of the investigation pursued in the analysis, till they terminate in the predicate, which was assumed in the analysis. This process will constitute the demonstration of the theorem.

If the assumption of the truth of the predicate in the analysis lead to some consequence which is inconsistent with any demonstrated truth, the false conclusion thus arrived at, indicates the falsehood of the predicate; and by reversing the process of the analysis, it may be demonstrated, that the theorem cannot be true.

It may here be remarked, that the geometrical analysis is more extensively useful in discovering the solution of problems than for investigating the demonstration of theorems.

From the nature of the subject, it must be at once obvious, that no general rules can be prescribed, which will be found applicable in all cases, and infallibly lead to the solution of every problem. The conditions of problems must suggest what constructions may be possible; and the consequences which follow from these constructions and the assumed solution, will shew the possibility or impossibility of arriving at some known property consistent with the data of the problem.

Though the data of a problem may be given in magnitude and position, certain ambiguities will arise, if they are not properly restricted. Two points may be considered as situated on the same side, or one on each side of a given line; and there may be two lines drawn from a given point making equal angles with a line given in position; and to avoid ambiguity, it must be stated on which side of the line the angle is to be formed.

A problem is said to be *determinate* when, with the prescribed conditions, it admits of one definite solution; the same construction which may be made on the other side of any given line, not being considered a different solution: and a problem is said to be *indeterminate* when it admits of more than one definite solution. This latter circumstance arises from the data not *absolutely fixing*, but *merely restricting* the quæsitæ, leaving certain points or lines not fixed in one position only. The number of given conditions may be insufficient for a single determinate solution; or relations may subsist among some of the given conditions from which one or more of the remaining given conditions may be deduced.

If the base of a right-angled triangle be given, and also the difference of the squares on the hypotenuse and perpendicular, the triangle is indeterminate. For though apparently here are three things given, the right angle, the base, and the difference of the squares on the hypotenuse and perpendicular, it is obvious that these three apparent conditions are in fact reducible to two: for since in a right-angled triangle, the sum of the squares on the base and on the perpendicular, is equal to the square on the hypotenuse, it follows that the difference of the squares on the hypotenuse and perpendicular, is equal to the square on the base of the triangle, and therefore the base is known from the difference of the squares on the hypotenuse and perpendicular being known. The conditions therefore are insufficient to determine a right-angled triangle; an indefinite number of triangles may be found with the prescribed conditions, whose vertices will lie in the line which is perpendicular to the base.

If a problem relate to the determination of a *single point*, and the data be sufficient to determine the position of that point, the problem is *determinate*: but if one or more of the conditions be omitted, the data which remain may be sufficient for the determination of more than one point, each of which satisfies the conditions of the problem; in that case, the problem is *indeterminate*: and in general, such points are found to be situated in some line, and hence such line is called the locus of the point which satisfies the conditions of the problem.

If any two given points *A* and *B* (fig. Euc. IV. 5.) be joined by a straight line *AB*, and this line be bisected in *D*, then if a perpendicular be drawn from the point of bisection, it is manifest that a circle

described with *any* point in the perpendicular as a center, and a radius equal to its distance from one of the given points, will pass through the other point, and the perpendicular will be the locus of all the circles which can be described passing through the two given points.

Again, if a third point *C* be taken, but not in the same straight line with the other two, and this point be joined with the first point *A*; then the perpendicular drawn from the bisection *E* of this line will be the locus of the centers of all circles which pass through the first and third points *A* and *C*. But the perpendicular at the bisection of the first and second points *A* and *B* is the locus of the centers of circles which pass through these two points. Hence the intersection *F* of these two perpendiculars, will be the center of a circle which passes through the three points and is called the intersection of the two loci. Sometimes this method of solving geometrical problems may be pursued with advantage, by constructing the locus of every two points separately, which are given in the conditions of the problem. In the Geometrical Exercises which follow, only those local problems are given where the locus is either a straight line or a circle.

Whenever the quæsitum is a point, the problem on being rendered indeterminate, becomes a locus, whether the deficient datum be of the essential or of the accidental kind. When the quæsitum is a straight line or a circle, (which were the only two loci admitted into the ancient Elementary Geometry) the problem *may* admit of an *accidentally indeterminate* case; but will not *invariably* or even very frequently do so. This will be the case, when the line or circle shall be ~~so~~ far arbitrary in its position, as depends upon the deficiency of a *single* condition to fix it perfectly;—that is, (for instance) one point in the line, or two points in the circle, may be determined from the given conditions, but the remaining one is indeterminate from the accidental relations among the data of the problem.

Determinate Problems become indeterminate by the merging of some one datum in the results of the remaining ones. This may arise in three different ways; first, from the coincidences of two points; secondly, from that of two straight lines; and thirdly, from that of two circles. These, further, are the only three ways in which this accidental coincidence of data can produce this indeterminateness; that is, in other words, convert the problem into a Porism.

In the original Greek of Euclid's Elements, the corollaries to the propositions are called porisms (πορίσματα); but this scarcely explains the nature of *porisms*, as it is manifest that they are different from simple deductions from the demonstrations of propositions. Some analogy, however, we may suppose them to have to the porisms or corollaries in the Elements. Pappus (Coll. Math. Lib. vii. pref.) informs us that Euclid wrote three books on Porisms. He defines "a porism to be something between a problem and a theorem, or that in which something is proposed to be investigated." Dr. Simson, to whom is due the merit of having restored the porisms of Euclid, gives the following definition of that class of propositions: "Porisma est propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem, datis, sed quæ ad ea quæ data sunt eandem habent relationem, convenire osten-

dendum est affectionem quandam communem in propositione descriptam." That is, "A Porism is a proposition in which it is proposed to demonstrate that some one thing, or more things than one, are given, to which, as also to each of innumerable other things, not given indeed, but which have the same relation to those which are given, it is to be shewn that there belongs some common affection described in the proposition." Professor Dugald Stewart defines a porism to be "A proposition affirming the possibility of finding one or more of the conditions of an indeterminate theorem." Professor Playfair in a paper (from which the following account is taken) on Porisms, printed in the Transactions of the Royal Society of Edinburgh, for the year 1792, defines a porism to be "A proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate or capable of innumerable solutions."

It may without much difficulty be perceived that this definition represents a porism as almost the same as an indeterminate problem. There is a large class of indeterminate problems which are, in general, loci, and satisfy certain defined conditions. Every indeterminate problem containing a locus may be made to assume the form of a porism, but not the converse. Porisms are of a more general nature than indeterminate problems which involve a locus.

The ancient geometers appear to have undertaken the solution of problems with a scrupulous and minute attention, which would scarcely allow any of the collateral truths to escape their observation. They never considered a problem as solved till they had distinguished all its varieties, and evolved separately every different case that could occur, carefully distinguishing whatever change might arise in the construction from any change that was supposed to take place among the magnitudes which were given. This cautious method of proceeding soon led them to see that there were circumstances in which the solution of a problem would cease to be possible; and this always happened when one of the conditions of the data was inconsistent with the rest. Such instances would occur in the simplest problems; but in the analysis of more complex problems, they must have remarked that their constructions failed, for a reason directly contrary to that assigned. Instances would be found where the lines, which, by their intersection, were to determine the thing sought, instead of intersecting one another, as they did in general, or of not meeting at all, would coincide with one another entirely, and consequently leave the question unresolved. The confusion thus arising would soon be cleared up, by observing, that a problem before determined by the intersection of two lines, would now become capable of an indefinite number of solutions. This was soon perceived to arise from one of the conditions of the problem involving another, or from two parts of the data becoming one, so that there was not left a sufficient number of independent conditions to confine the problem to a single solution, or any determinate number of solutions. It was not difficult afterwards to perceive, that these cases of problems formed very curious propositions, of an indeterminate nature between problems and theorems, and that they admitted of being enunciated separately. It was to such propositions so enunciated that the ancient geometers gave the name of *Porisms*.

Besides, it will be found, that some problems are possible within

certain limits, and that certain magnitudes increase while others decrease within those limits; and after having reached a certain value, the former begin to decrease, while the latter increase. This circumstance gives rise to questions of *maxima* and *minima*, or the greatest and least values which certain magnitudes may admit of in indeterminate problems.

In the following collection of problems and theorems, most will be found to be of so simple a character, (being almost obvious deductions from propositions in the Elements) as scarcely to admit of the principle of the Geometrical Analysis being applied, in their solution.

It must however be recollected that a clear and exact knowledge of the first principles of Geometry must necessarily precede any intelligent application of them. Indistinctness or defectiveness of understanding with respect to these, will be a perpetual source of error and confusion. The learner is therefore recommended to understand the principles of the Science, and their connexion, fully, before he attempt any applications of them. The following directions may assist him in his proceedings.

#### ANALYSIS OF THEOREMS.

1. Assume that the Theorem is true.
2. Proceed to examine any consequences that result from this admission, by the aid of other truths respecting the diagram, which have been already proved.
3. Examine whether any of these consequences are already known to be *true*, or to be *false*.
4. If any one of them be false, we have arrived at a *reductio ad absurdum*, which proves that the theorem itself is false, as in Euc. I. 25.
5. If none of the consequences so deduced be *known* to be either true or false, proceed to deduce other consequences from all or any of these, as in (2).
6. Examine these results, and proceed as in (3) and (4); and if still without any conclusive indications of the truth or falsehood of the alleged theorem, proceed still further, until such are obtained.

#### ANALYSIS OF PROBLEMS.

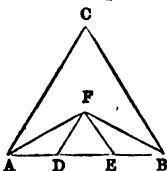
1. In general, any given problem will be found to depend on several problems and theorems, and these ultimately on some problem or theorem in Euclid.
2. Describe the diagram as directed in the enunciation, and suppose the solution of the problem effected.
3. Examine the relations of the lines, angles, triangles, &c. in the diagram, and find the dependence of the assumed solution on some theorem or problem in the Elements.
4. If such cannot be found, draw other lines parallel or perpendicular as the case may require, join given points, or points assumed in the solution, and describe circles if need be: and then proceed to trace the dependence of the assumed solution on some theorem or problem in Euclid.
5. Let not the first unsuccessful attempts at the solution of a Problem be considered as of no value; such attempts have been found to lead to the discovery of other theorems and problems.

## GEOMETRICAL EXERCISES ON BOOK I.

### PROPOSITION I. PROBLEM.

*To trisect a given straight line.*

**ANALYSIS.** Let  $AB$  be the given straight line, and suppose it divided into three equal parts in the points  $D, E$ .



On  $DE$  describe an equilateral triangle  $DEF$ ,  
then  $DF$  is equal to  $AD$ , and  $FE$  to  $EB$ .  
On  $AB$  describe an equilateral triangle  $ABC$ ,  
and join  $AF, FB$ .

Then because  $AD$  is equal to  $DF$ ,  
therefore the angle  $AFD$  is equal to the angle  $DAF$ ,  
and the two angles  $DAF, DFA$  are double of one of them  $DAF$ .  
But the angle  $FDE$  is equal to the angles  $DAF, DFA$ ,  
and the angle  $FDE$  is equal to  $DAC$ , each being an angle of an  
equilateral triangle;

therefore the angle  $DAC$  is double the angle  $DAF$ ;  
wherefore the angle  $DAC$  is bisected by  $AF$ .

Also because the angle  $FAC$  is equal to the angle  $FAD$ ,  
and the angle  $FAD$  to  $DFA$ ;  
therefore the angle  $CAF$  is equal to the alternate angle  $AFD$ :  
and consequently  $FD$  is parallel to  $AC$ .

**Synthesis.** Upon  $AB$  describe an equilateral triangle  $ABC$ ,  
bisect the angles at  $A$  and  $B$  by the straight lines  $AF, BF$ , meeting in  $F$ ;  
through  $F$  draw  $FD$  parallel to  $AC$ , and  $FE$  parallel to  $BC$ .

Then  $AB$  is trisected in the points  $D, E$ .

For since  $AC$  is parallel to  $FD$  and  $FA$  meets them,  
therefore the alternate angles  $FAC, AFD$  are equal;  
but the angle  $FAD$  is equal to the angle  $FAC$ ,  
hence the angle  $DAF$  is equal to the angle  $AFD$ ,  
and therefore  $DF$  is equal to  $DA$ .

But the angle  $FDE$  is equal to the angle  $CAB$ ,  
and  $FED$  to  $CBA$ ; (I. 29.)

therefore the remaining angle  $DFE$  is equal to the remaining angle  
 $ACB$ .

Hence the three sides of the triangle  $DFE$  are equal to one another,  
and  $DF$  has been shewn to be equal to  $DA$ ,  
therefore  $AD, DE, EB$  are equal to one another.

Hence the following theorem.

If the angles at the base of an equilateral triangle be bisected by  
two lines which meet at a point within the triangle; the two lines  
drawn from this point parallel to the sides of the triangle, divide the  
base into three equal parts.

**Note.** There is another method whereby a line may be divided into three equal parts:—by drawing from one extremity of the given line, another making an acute angle with it, and taking three equal distances from the extremity, then joining the extremities, and through the other two points of division, drawing lines parallel to this line through the other two points of division, and to the given line; the three triangles thus formed are equal in all respects. This may be extended for any number of parts, and is a particular case of Euc. VI. 10.

### PROPOSITION II. THEOREM.

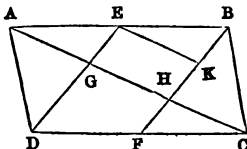
*If two opposite sides of a parallelogram be bisected, and two lines be drawn from the points of bisection to the opposite angles, these two lines trisect the diagonal.*

Let  $ABCD$  be a parallelogram of which the diagonal is  $AC$ .

Let  $AB$  be bisected in  $E$ , and  $DC$  in  $F$ ,

also let  $DE$ ,  $FB$  be joined cutting the diagonal in  $G$ ,  $H$ .

Then  $AC$  is trisected in the points  $G$ ,  $H$ .



Through  $E$  draw  $EK$  parallel to  $AC$  and meeting  $FB$  in  $K$ .

Then because  $EB$  is the half of  $AB$ , and  $DF$  the half of  $DC$ ;  
therefore  $EB$  is equal to  $DF$ ;

and these equal and parallel straight lines are joined towards the same parts by  $DE$  and  $FB$ ;

therefore  $DE$  and  $FB$  are equal and parallel. (I. 33.)

And because  $AEB$  meets the parallels  $EK$ ,  $AC$ ,

therefore the exterior angle  $BEK$  is equal to the interior angle  $EAG$ .

For a similar reason, the angle  $EBK$  is equal to the angle  $AEG$ .

Hence in the triangles  $AEG$ ,  $EBK$ , there are the two angles  $GAE$ ,  $AEG$  in the one, equal to the two angles  $KEB$ ,  $EBK$  in the other, and one side adjacent to the equal angles in each triangle, namely  $AE$  equal to  $EB$ ;

therefore  $AG$  is equal to  $EK$ , (I. 26.)

but  $EK$  is equal to  $GH$ , (I. 34.) therefore  $AG$  is equal to  $GH$ .

By a similar process, it may be shewn that  $GH$  is equal to  $HC$ .

Hence  $AG$ ,  $GH$ ,  $HC$  are equal to one another,

and therefore  $AC$  is trisected in the points  $G$ ,  $H$ .

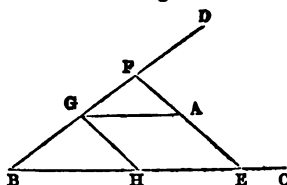
It may also be proved that  $BF$  is trisected in  $H$  and  $K$ .

### PROPOSITION III. PROBLEM.

*Draw through a given point, between two straight lines not parallel, a straight line which shall be bisected in that point.*

**Analysis.** Let  $BC$ ,  $BD$  be the two lines meeting in  $B$ , and let  $A$  be the given point between them.

Suppose the line  $EA$  drawn through  $A$ , so that  $EA$  is equal to  $AF$ ;



through  $A$  draw  $AG$  parallel to  $BC$ , and  $GH$  parallel to  $EF$ .

Then  $AGHE$  is a parallelogram, wherefore  $AE$  is equal to  $GH$ , but  $EA$  is equal to  $AF$  by hypothesis; therefore  $GH$  is equal to  $AF$ .

Hence in the triangles  $BHG$ ,  $GAF$ , the angles  $HBG$ ,  $AGF$  are equal, as also  $BGH$ ,  $GFA$ , (I. 29.)

also the side  $GH$  is equal to  $AF$ ;

whence the other parts of the triangles are equal, (I. 26.)

therefore  $BG$  is equal to  $GF$ .

Synthesis. Through the given point  $A$ , draw  $AG$  parallel to  $BC$ ;

on  $GD$ , take  $GF$  equal to  $GB$ ;

then  $F$  is a second point in the required line:

join the points  $F$ ,  $A$ , and produce  $FA$  to meet  $BC$  in  $E$ ;

then the line  $FE$  is bisected in the point  $A$ ;

draw  $GH$  parallel to  $AE$ .

Then in the triangles  $BGH$ ,  $GFA$ , the side  $BG$  is equal to  $GF$ , and the angles  $GBH$ ,  $BGH$  are respectively equal to  $FGA$ ,  $GFA$ ;

wherefore  $GH$  is equal to  $AF$ , (I. 26.)

but  $GH$  is equal to  $AE$ , (I. 34.)

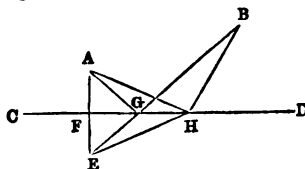
therefore  $AE$  is equal to  $AF$ , or  $EF$  is bisected in  $A$ .

#### PROPOSITION IV. PROBLEM.

*From two given points on the same side of a straight line given in position, draw two straight lines which shall meet in that line, and make equal angles with it; also prove, that the sum of these two lines is less than the sum of any other two lines drawn to any other point in the line.*

Analysis. Let  $A$ ,  $B$  be the two given points, and  $CD$  the given line.

Suppose  $G$  the required point in the line, such that  $AG$  and  $BG$  being joined, the angle  $AGC$  is equal to the angle  $BGD$ .



Draw  $AF$  perpendicular to  $CD$  and meeting  $BG$  produced in  $E$ .

Then, because the angle  $BGD$  is equal to  $AGF$ , (hyp.)

and also to the vertical angle  $FGE$ , (I. 15.)

therefore the angle  $AGF$  is equal to the angle  $EGF$ ;



also the right angle  $AFG$  is equal to the right angle  $EFG$ ,  
and the side  $FG$  is common to the two triangles  $AFG$ ,  $EFG$ ,  
therefore  $AG$  is equal to  $EG$ , and  $AF$  to  $FE$ .

Hence the point  $E$  being known, the point  $G$  is determined by the intersection of  $CD$  and  $BE$ .

Synthesis. From  $A$  draw  $AF$  perpendicular to  $CD$ , and produce it to  $E$ , making  $FE$  equal to  $AF$ , and join  $BE$  cutting  $CD$  in  $G$ .

Join also  $AG$ .

Then  $AG$  and  $BG$  make equal angles with  $CD$ .

For since  $AF$  is equal to  $FE$ , and  $FG$  is common to the two triangles  $AGF$ ,  $EGF$ , and the included angles  $AFG$ ,  $EFG$  are equal;

therefore the base  $AG$  is equal to the base  $EG$ ,

and the angle  $AGF$  to the angle  $EGF$ ;

but the angle  $EGF$  is equal to the vertical angle  $BGD$ ,

therefore the angle  $AGF$  is equal to the angle  $BGD$ ;

that is, the straight lines  $AG$  and  $BG$  make equal angles with the straight line  $CD$ .

Also the sum of the lines  $AG$ ,  $GB$  is a minimum.

For take any other point  $H$  in  $CD$ , and join  $EH$ ,  $HB$ ,  $AH$ .

Then since any two sides of a triangle are greater than the third side, therefore  $EH$ ,  $HB$  are greater than  $EB$  in the triangle  $EBH$ .

But  $EG$  is equal to  $AG$ , and  $EH$  to  $AH$ ;

therefore  $AH$ ,  $HB$  are greater than  $AG$ ,  $GB$ .

That is,  $AG$ ,  $GB$  are less than any other two lines which can be drawn from  $A$ ,  $B$ , to any other point  $H$  in the line  $CD$ .

By means of this Proposition may be found the shortest path from one given point to another, subject to the condition, that it shall meet two given lines.

#### PROPOSITION V. PROBLEM.

*Given one angle, a side opposite to it, and the sum of the other two sides, construct the triangle.*

Analysis. Suppose  $BAC$  the triangle required, having  $BC$  equal to the given side,  $BAC$  equal to the given angle opposite to  $BC$ , also  $BD$  equal to the sum of the other two sides.



Join  $DC$ .

Then since the two sides  $BA$ ,  $AC$  are equal to  $BD$ , by taking  $BA$  from these equals, the remainder  $AC$  is equal to the remainder  $AD$ .

Hence the triangle  $ADC$  is isosceles, and therefore the angle  $ADC$  is equal to the angle  $ACD$ .

But the exterior angle  $BAC$  of the triangle  $ADC$  is equal to the two interior and opposite angles  $ACD$  and  $ADC$ :

Wherefore the angle  $BAC$  is double the angle  $BDC$ , and  $BDC$  is the half of the angle  $BAC$ .

Hence the synthesis.

At the point  $D$  in  $BD$ , make the angle  $BDC$  equal to half the given angle,

and from  $B$  the other extremity of  $BD$ , draw  $BC$  equal to the given side, and meeting  $DC$  in  $C$ ,

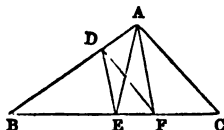
at  $C$  in  $CD$  make the angle  $DCA$  equal to the angle  $CDA$ , so that  $CA$  may meet  $BD$  in the point  $A$ .

Then the triangle  $ABC$  shall have the required conditions.

#### PROPOSITION VI. PROBLEM.

*To bisect a triangle by a line drawn from a given point in one of the sides.*

Analysis. Let  $ABC$  be the given triangle, and  $D$  the given point in the side  $AB$ .



Suppose  $DF$  the line drawn from  $D$  which bisects the triangle; therefore the triangle  $DBF$  is half of the triangle  $ABC$ .

Bisect  $BC$  in  $E$ , and join  $AE$ ,  $DE$ ,  $AF$ ,

then the triangle  $ABE$  is half of the triangle  $ABC$ :

hence the triangle  $ABE$  is equal to the triangle  $DBF$ ;

take away from these equals the triangle  $DDE$ ,

therefore the remainder  $ADE$  is equal to the remainder  $DEF$ .

But  $ADE$ ,  $DEF$  are equal triangles upon the same base  $DE$ , and on the same side of it,

they are therefore between the same parallels, (I. 39.)

that is,  $AF$  is parallel to  $DE$ ,

therefore the point  $F$  is determined.

Synthesis. Bisect the base  $BC$  in  $E$ , join  $DE$ ,

from  $A$ , draw  $AF$  parallel to  $DE$ , and join  $DF$ .

Then because  $DE$  is parallel to  $AF$ ,

therefore the triangle  $ADE$  is equal to the triangle  $DEF$ ; (I. 37.)

to each of these equals, add the triangle  $BDE$ ,

therefore the whole triangle  $ABE$  is equal to the whole  $DBF$ ,

but  $ABE$  is half of the whole triangle  $ABC$ ;

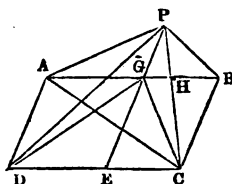
therefore  $DBF$  is also half of the triangle  $ABC$ .

#### PROPOSITION VII. THEOREM.

*If from a point without a parallelogram lines be drawn to the extremities of two adjacent sides, and of the diagonal which they include; of the triangles thus formed, that, whose base is the diagonal, is equal to the sum of the other two.*

Let  $ABCD$  be a parallelogram of which  $AC$  is one of the diagonals, and let  $P$  be any point without it: and let  $AP$ ,  $PC$ ,  $BP$ ,  $PD$  be joined.

Then the triangles  $APD$ ,  $APB$  are together equivalent to the triangle  $APC$ .



Draw  $PGE$  parallel to  $AD$  or  $BC$ , and meeting  $AB$  in  $G$ , and  $DC$  in  $E$ ; and join  $DG$ ,  $GC$ .

Then the triangles  $CBP$ ,  $CBG$  are equal: (I. 37.)

and taking the common part  $CBH$  from each,  
the remainders  $PHB$ ,  $CHG$  are equal.

Again, the triangles  $DAP$ ,  $DAG$  are equal; (I. 37.)

also the triangles  $DAG$ ,  $AGC$  are equal, being on the same base  $AG$ , and between the same parallels  $AG$ ,  $DC$ :

therefore the triangle  $DAP$  is equal to the triangle  $AGC$ :

but the triangle  $PHB$  is equal to the triangle  $CHG$ ,

wherefore the triangles  $PHB$ ,  $DAP$  are equal to  $AGC$ ,  $CHG$ , or

$ACH$ , add to these equals the triangle  $APH$ ,

therefore the triangles  $APH$ ,  $PHB$ ,  $DAP$  are equal to  $APH$ ,  $ACH$ ,  
that is, the triangles  $APB$ ,  $DAP$  are together equal to the triangle  $PAC$ .

If the point  $P$  be within the parallelogram, then the *difference* of the triangles  $APB$ ,  $DAP$  may be proved to be equal to the triangle  $PAC$ .

# I.

8. Describe an isosceles triangle upon a given base and having each of the sides double of the base, without using any proposition of the Elements subsequent to the first three. If the base and sides be given, what condition must be fulfilled with regard to the magnitude of each of the equal sides in order that an isosceles triangle may be constructed?

9. In the fig. Euc. I. 5. If  $FC$  and  $BG$  meet in  $H$ , then prove that  $AH$  bisects the angle  $BAC$ .

10. In the fig. Euc. I. 5. If the angle  $FBG$  be equal to the angle  $ABC$ , and  $BG$ ,  $CF$ , intersect in  $O$ ; the angle  $BOF$  is equal to twice the angle  $BAC$ .

11. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides, the angles made by them with the base are each equal to half the vertical angle.

12. A line drawn bisecting the angle contained by the two equal sides of an isosceles triangle, bisects the third side at right angles.

13. If a straight line drawn bisecting the vertical angle of a triangle also bisect the base, the triangle is isosceles.

14. Given two points one on each side of a given straight line; find a point in the line such that the angle contained by two lines drawn to the given points may be bisected by the given line.

15. In the fig. Euc. I. 5, let  $F$  and  $G$  be the points in the sides  $AB$  and  $AC$  produced, and let lines  $FH$  and  $GK$  be drawn perpendicular and equal to  $FC$  and  $GB$  respectively: also if  $BH$ ,  $CK$ , or these lines produced meet in  $O$ ; prove that  $BH$  is equal to  $CK$ , and  $BO$  to  $CO$ .

16. From every point of a given straight line, the straight lines drawn to each of two given points on opposite sides of the line are equal: prove that the line joining the given points will cut the given line at right angles.

17. If  $A$  be the vertex of an isosceles triangle  $ABC$ , and  $BA$  be produced so that  $AD$  is equal to  $BA$ , and  $DC$  be drawn; shew that  $BCD$  is a right angle.

18. The straight line  $EDF$ , drawn at right angles to  $BC$  the base of an isosceles triangle  $ABC$ , cuts the side  $AB$  in  $D$ , and  $CA$  produced in  $E$ ; shew that  $AED$  is an isosceles triangle.

19. In the fig. Euc. I. 1, if  $AB$  be produced both ways to meet the circles in  $D$  and  $E$ , and from  $C$ ,  $CD$  and  $CE$  be drawn; the figure  $CDE$  is an isosceles triangle having each of the angles at the base, equal to one fourth of the angle at the vertex of the triangle.

20. From a given point, draw two straight lines making equal angles with two given straight lines intersecting one another.

21. From a given point to draw a straight line to a given straight line, that shall be bisected by another given straight line.

22. Place a straight line of given length between two given straight lines which meet, so that it shall be equally inclined to each of them.

23. To determine that point in a straight line from which the straight lines drawn to two other given points shall be equal, provided the line joining the two given points is not perpendicular to the given line.

24. In a given straight line to find a point equally distant from two given straight lines. In what case is this impossible?

25. If a line intercepted between the extremity of the base of an isosceles triangle, and the opposite side (produced if necessary) be equal to a side of the triangle, the angle formed by this line and the base produced, is equal to three times either of the equal angles of the triangle.

26. In the base  $BC$  of an isosceles triangle  $ABC$ , take a point  $D$ , and in  $CA$  take  $CE$  equal to  $CD$ , let  $ED$  produced meet  $AB$  produced in  $F$ ; then  $3.AEF = 2$  right angles  $+ AFE$ , or  $= 4$  right angles  $+ AFE$ .

27. If from the base to the opposite sides of an isosceles triangle, three straight lines be drawn, making equal angles with the base, viz. one from its extremity, the other two from any other point in it, these two shall be together equal to the first.

28. A straight line is drawn, terminated by one of the sides of an isosceles triangle, and by the other side produced, and bisected by the base; prove that the straight lines, thus intercepted between the

vertex of the isosceles triangle, and this straight line, are together equal to the two equal sides of the triangle.

29. In a triangle, if the lines bisecting the angles at the base be equal, the triangle is isosceles, and the angle contained by the bisecting lines is equal to an exterior angle at the base of the triangle.

30. In a triangle, if lines be equal when drawn from the extremities of the base, (1) perpendicular to the sides, (2) bisecting the sides, (3) making equal angles with the sides; the triangle is isosceles: and then these lines which respectively join the intersections of the sides, are parallel to the base.

## II.

31.  $ABC$  is a triangle right-angled at  $B$ , and having the angle  $A$  double the angle  $C$ ; shew that the side  $BC$  is less than double the side  $AB$ .

32. If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

33. If from the right angle of a right-angled triangle, two straight lines be drawn, one perpendicular to the base, and the other bisecting it, they will contain an angle equal to the difference of the two acute angles of the triangle.

34. If the vertical angle  $CAB$  of a triangle  $ABC$  be bisected by  $AD$ , to which the perpendiculars  $CE$ ,  $BF$  are drawn from the remaining angles: bisect the base  $BC$  in  $G$ , join  $GE$ ,  $GF$ , and prove these lines equal to each other.

35. The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.

36. If one angle at the base of a triangle be double of the other, the less side is equal to the sum or difference of the segments of the base made by the perpendicular from the vertex, according as the angle is greater or less than a right angle.

37. If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines, a line be drawn to the opposite angle of the triangle, it will bisect that angle.

38. From the vertex of a scalene triangle draw a right line to the base, which shall exceed the less side as much as it is exceeded by the greater.

39. Divide a right angle into three equal angles.

40. One of the acute angles of a right-angled triangle is three times as great as the other; trisect the smaller of these.

41. Prove that the sum of the distances of any point within a triangle from the three angles is greater than half the perimeter of the triangle.

42. The perimeter of an isosceles triangle is less than that of any other equal triangle upon the same base.

43. If from the angles of a triangle  $ABC$ , straight lines  $ADE$ ,  $BDF$ ,  $CDG$  be drawn through a point  $D$  to the opposite sides, prove that the sides of the triangle are together greater than the three

lines drawn to the point  $D$ , and less than twice the same, but greater than two-thirds of the lines drawn through the point to the opposite sides.

44. In a plane triangle an angle is right, acute or obtuse, according as the line joining the vertex of the angle with the middle point of the opposite side is equal to, greater or less than half of that side.

45. If the straight line  $AD$  bisect the angle  $A$  of the triangle  $ABC$ , and  $BDE$  be drawn perpendicular to  $AD$  and meeting  $AC$  or  $AC$  produced in  $E$ , shew that  $BD = DE$ .

46. The side  $BC$  of a triangle  $ABC$  is produced to a point  $D$ . The angle  $ACB$  is bisected by a line  $CE$  which meets  $AB$  in  $E$ . A line is drawn through  $E$  parallel to  $BC$  and meeting  $AC$  in  $F$ , and the line bisecting the exterior angle  $ACD$ , in  $G$ . Shew that  $EF$  is equal to  $FG$ .

47. The sides  $AB$ ,  $AC$ , of a triangle are bisected in  $D$  and  $E$  respectively, and  $BE$ ,  $CD$ , are produced until  $EF = EB$ , and  $GD = DC$ ; shew that the line  $GF$  passes through  $A$ .

48. In a triangle  $ABC$ ,  $AD$  being drawn perpendicular to the straight line  $BD$  which bisects the angle  $B$ , shew that a line drawn from  $D$  parallel to  $BC$  will bisect  $AC$ .

49. If the sides of a triangle be trisected and lines be drawn through the points of section adjacent to each angle so as to form another triangle, this shall be in all respects equal to the first triangle.

50. Between two given straight lines it is required to draw a straight line which shall be equal to one given straight line, and parallel to another.

51. If from the vertical angle of a triangle three straight lines be drawn, one bisecting the angle, another bisecting the base, and the third perpendicular to the base, the first is always intermediate in magnitude and position to the other two.

52. In the base of a triangle, find the point from which, lines drawn parallel to the sides of the triangle and limited by them, are equal.

53. In the base of a triangle, to find a point from which if two lines be drawn, (1) perpendicular, (2) parallel, to the two sides of the triangle, their sum shall be equal to a given line.

### III.

54. In the figure of Euc. I. 1, the given line is produced to meet either of the circles in  $P$ ; shew that  $P$  and the points of intersection of the circles, are the angular points of an equilateral triangle.

55. If each of the equal angles of an isosceles triangle be one-fourth of the third angle, and from one of them a line be drawn at right angles to the base meeting the opposite side produced; then will the part produced, the perpendicular, and the remaining side, form an equilateral triangle.

56. In the figure Euc. I. 1, if the sides  $CA$ ,  $CB$  of the equilateral triangle  $ABC$  be produced to meet the circles in  $F$ ,  $G$ , respectively, and if  $C'$  be the point in which the circles cut one another on the

other side of  $AB$ : prove the points  $F, C, G$  to be in the same straight line; and the figure  $CFG$  to be an equilateral triangle.

57.  $ABC$  is a triangle and the exterior angles at  $B$  and  $C$  are bisected by lines  $BD, CD$  respectively, meeting in  $D$ : shew that the angle  $BDC$  and half the angle  $BAC$  make up a right angle.

58. If the exterior angle of a triangle be bisected, and the angles of the triangle made by the bisectors be bisected, and so on, the triangles so formed will tend to become eventually equilateral.

59. If in the three sides  $AB, BC, CA$  of an equilateral triangle  $ABC$ , distances  $AE, BF, CG$  be taken, each equal to a third of one of the sides, and the points  $E, F, G$  be respectively joined (1) with each other, (2) with the opposite angles: shew that the two triangles so formed, are equilateral triangles.

#### IV.

60. Describe a right-angled triangle upon a given base, having given also the perpendicular from the right angle upon the hypotenuse.

61. Given one side of a right-angled triangle, and the difference between the hypotenuse and the sum of the other two sides, to construct the triangle.

62. Construct an isosceles right-angled triangle, having given (1) the sum of the hypotenuse and one side; (2) their difference.

63. Describe a right-angled triangle of which the hypotenuse and the difference between the other two sides are given.

64. Given the base of an isosceles triangle, and the sum or difference of a side and the perpendicular from the vertex on the base. Construct the triangle.

65. Make an isosceles triangle of given altitude whose sides shall pass through two given points and have its base on a given straight line.

66. Construct an equilateral triangle, having given the length of the perpendicular drawn from one of the angles on the opposite side.

67. Having given the straight lines which bisect the angles at the base of an equilateral triangle, determine a side of the triangle.

68. Having given two sides and an angle of a triangle, construct the triangle, distinguishing the different cases.

69. Having given the base of a triangle, the difference of the sides, and the difference of the angles at the base; to describe the triangle.

70. Given the perimeter and the angles of a triangle, to construct it.

71. Having given the base of a triangle, and half the sum and half the difference of the angles at the base; to construct the triangle.

72. Having given two lines, which are not parallel, and a point between them; describe a triangle having two of its angles in the respective lines, and the third at the given point; and such that the sides shall be equally inclined to the lines which they meet.

73. Construct a triangle, having given the three lines drawn from the angles to bisect the sides opposite.

74. Given one of the angles at the base of a triangle, the base itself, and the sum of the two remaining sides, to construct the triangle.

75. Given the base, an angle adjacent to the base, and the difference of the sides of a triangle, to construct it.

76. Given one angle, a side opposite to it, and the difference of the other two sides; to construct the triangle.

77. Given the base and the sum of the two other sides of a triangle, construct it so that the line which bisects the vertical angle shall be parallel to a given line.

## V.

78. From a given point without a given straight line, to draw a line making an angle with the given line equal to a given rectilineal angle.

79. Through a given point  $A$ , draw a straight line  $ABC$  meeting two given parallel straight lines in  $B$  and  $C$ , such that  $BC$  may be equal to a given straight line.

80. If the line joining two parallel lines be bisected, all the lines drawn through the point of bisection and terminated by the parallel lines are also bisected in that point.

81. Three given straight lines issue from a point: draw another straight line cutting them so that the two segments of it intercepted between them may be equal to one another.

82.  $AB, AC$  are two straight lines,  $B$  and  $C$  given points in the same;  $BD$  is drawn perpendicular to  $AC$ , and  $DE$  perpendicular to  $AB$ ; in like manner  $CF$  is drawn perpendicular to  $AB$ , and  $FG$  to  $AC$ . Shew that  $EG$  is parallel to  $BC$ .

83.  $ABC$  is a right-angled triangle, and the sides  $AC, AB$  are produced to  $D$  and  $F$ ; bisect  $FBC$  and  $BCD$  by the lines  $BE, CE$ , and from  $E$  let fall the perpendiculars  $EF, ED$ . Prove (without assuming any properties of parallels) that  $ADEF$  is a square.

84. Two pairs of equal straight lines being given, shew how to construct with them the greatest parallelogram.

85. With two given lines as diagonals describe a parallelogram which shall have an angle equal to a given angle. Within what limits must the given angle lie?

86. Having given one of the diagonals of a parallelogram, the sum of the two adjacent sides and the angle between them, construct the parallelogram.

87. One of the diagonals of a parallelogram being given, and the angle which it makes with one of the sides, complete the parallelogram, so that the other diagonal may be parallel to a given line.

88.  $ABCD, A'B'C'D'$  are two parallelograms whose corresponding sides are equal, but the angle  $A$  is greater than the angle  $A'$ , prove that the diameter  $AC$  is less than  $A'C'$ , but  $BD$  greater than  $B'D'$ .

89. If in the diagonal of a parallelogram any two points equidistant from its extremities be joined with the opposite angles, a figure will be formed which is also a parallelogram.

90. From each angle of a parallelogram a line is drawn making



the same angle towards the same parts with an adjacent side, taken always in the same order; shew that these lines form another parallelogram *similar* to the original one.

91. Along the sides of a parallelogram taken in order, measure  $AA' = BB' = CC' = DD'$ : the figure  $A'B'C'D'$  will be a parallelogram.

92. On the sides  $AB, BC, CD, DA$ , of a parallelogram, set off  $AE, BF, CG, DH$ , equal to each other, and join  $AF, BG, CH, DE$ : these lines form a parallelogram, and the difference of the angles  $AFB, BGC$ , equals the difference of any two proximate angles of the two parallelograms.

93.  $OB, OC$  are two straight lines at right angles to each other, through any point  $P$  any two straight lines are drawn intersecting  $OB, OC$ , in  $B, B', C, C'$ , respectively. If  $D$  and  $D'$  be the middle points of  $BB'$  and  $CC'$ , shew that the angle  $B'PD'$  is equal to the angle  $DOD$ .

94.  $ABCD$  is a parallelogram of which the angle  $C$  is opposite to the angle  $A$ . If through  $A$  any straight line be drawn, then the distance of  $C$  is equal to the sum or difference of the distances of  $B$  and of  $D$  from that straight line, according as it lies without or within the parallelogram.

95. Upon stretching two chains  $AC, BD$ , across a field  $ABCD$ , I find that  $BD$  and  $AC$  make equal angles with  $DC$ , and that  $AC$  makes the same angle with  $AD$  that  $BD$  does with  $BC$ ; hence prove that  $AB$  is parallel to  $CD$ .

96. To find a point in the side or side produced of any parallelogram, such that the angle it makes with the line joining the point and one extremity of the opposite side, may be bisected by the line joining it with the other extremity.

97. When the corner of the leaf of a book is turned down a second time, so that the lines of folding are parallel and equidistant, the space in the second fold is equal to three times that in the first.

## VI.

98. If the points of bisection of the sides of a triangle be joined, the triangle so formed shall be one-fourth of the given triangle.

99. If in the triangle  $ABC$ ,  $BC$  be bisected in  $D$ ,  $AD$  joined and bisected in  $E$ ,  $BE$  joined and bisected in  $F$ , and  $CF$  joined and bisected in  $G$ ; then the triangle  $EFG$  will be equal to one-eighth of the triangle  $ABC$ .

100. Shew that the areas of the two equilateral triangles in Prob. 59, p. 78, are respectively, one-third and one-seventh of the area of the original triangle.

101. To describe a triangle equal to a given triangle, (1) when the base, (2) when the altitude of the required triangle is given.

102. To describe a triangle equal to the sum or difference of two given triangles.

103. Upon a given base describe an isosceles triangle equal to a given triangle.

104. Describe a right-angled triangle equal to a given triangle  $ABC$ .

105. To a given straight line apply a triangle which shall be equal

to a given parallelogram and have one of its angles equal to a given rectilineal angle.

106. Transform a given rectilineal figure into a triangle whose vertex shall be in a given angle of the figure, and whose base shall be in one of the sides.

107. Divide a triangle by two straight lines into three parts which when properly arranged shall form a parallelogram whose angles are of a given magnitude.

108. Shew that a scalene triangle cannot be divided into two parts which will coincide.

109. If two sides of a triangle be given, the triangle will be greatest when they contain a right angle.

110. Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in the given point.

111. Of all triangles having the same base and the same perimeter, that is the greatest which has the two undetermined sides equal.

112. Divide a triangle into three equal parts, (1) by lines drawn from a point in one of the sides: (2) by lines drawn from the angles to a point within the triangle: (3) by lines drawn from a given point within the triangle. In how many ways can the third case be done?

113. Divide an equilateral triangle into nine equal parts.

114. Bisect a parallelogram, (1) by a line drawn from a point in one of its sides: (2) by a line drawn from a given point within or without it: (3) by a line perpendicular to one of the sides: (4) by a line drawn parallel to a given line.

115. From a given point in one side produced of a parallelogram, draw a straight line which shall divide the parallelogram into two equal parts.

116. To trisect a parallelogram by lines drawn (1) from a given point in one of its sides, (2) from one of its angular points.

## VII.

117. To describe a rhombus which shall be equal to any given quadrilateral figure.

118. Describe a parallelogram which shall be equal in area and perimeter to a given triangle.

119. Find a point in the diagonal of a square produced, from which if a straight line be drawn parallel to any side of the square, and meeting another side produced, it will form together with the produced diagonal and produced side, a triangle equal to the square.

120. If from any point within a parallelogram, straight lines be drawn to the angles, the parallelogram shall be divided into four triangles, of which each two opposite are together equal to one-half of the parallelogram.

121. If  $ABCD$  be a parallelogram, and  $E$  any point in the diagonal  $AC$ , or  $AC$  produced; shew that the triangles  $EBC$ ,  $EDC$ , are equal, as also the triangles  $EBA$  and  $EBD$ .

122.  $ABCD$  is a parallelogram, draw  $DFG$  meeting  $BC$  in  $F$ ,

and  $AB$  produced in  $G$ ; join  $AF$ ,  $CG$ ; then will the triangles  $ABF$ ,  $CFG$  be equal to one another.

123.  $ABCD$  is a parallelogram,  $E$  the point of intersection of its diagonals, and  $K$  any point in  $AD$ . If  $KB$ ,  $KC$  be joined, shew that the figure  $BKEC$  is one-fourth of the parallelogram.

124. Let  $ABCD$  be a parallelogram, and  $O$  any point within it, through  $O$  draw lines parallel to the sides of  $ABCD$ , and join  $OA$ ,  $OC$ ; prove that the difference of the parallelograms  $DO$ ,  $BO$  is twice the triangle  $OAC$ .

125. The diagonals  $AC$ ,  $BD$  of a parallelogram intersect in  $O$ , and  $P$  is a point within the triangle  $AOB$ ; prove that the difference of the triangles  $APB$ ,  $CPD$  is equal to the sum of the triangles  $APC$ ,  $BPD$ .

126. If  $K$  be the common angular point of the parallelograms about the diameter  $AC$  (fig. Euc. I. 43.) and  $BD$  be the other diameter, the difference of these parallelograms is equal to twice the triangle  $BKD$ .

127. The perimeter of a square is less than that of any other parallelogram of equal area.

128. Shew that of all equiangular parallelograms of equal perimeters, that which is equilateral is the greatest.

129. Prove that the perimeter of an isosceles triangle is greater than that of an equal right-angled parallelogram of the same altitude.

### VIII.

130. If a quadrilateral figure is bisected by one diagonal, the second diagonal is bisected by the first.

131. If two opposite angles of a quadrilateral figure are equal, shew that the angles between opposite sides produced are equal.

132. Prove that the sides of any four-sided rectilinear figure are together greater than the two diagonals.

133. The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles, from any point within the figure, except their intersection.

134. The longest side of a given quadrilateral is opposite to the shortest; shew that the angles adjacent to the shortest side are together greater than the sum of the angles adjacent to the longest side.

135. Give any two points in the opposite sides of a trapezium, inscribe in it a parallelogram having two of its angles at these points.

136. Shew that in every quadrilateral plane figure, two parallelograms can be described upon two opposite sides as diagonals, such that the other two diagonals shall be in the same straight line and equal.

137. Describe a quadrilateral figure whose sides shall be equal to four given straight lines. What limitation is necessary?

138. If the sides of a quadrilateral figure be bisected and the points of bisection joined, the included figure is a parallelogram, and equal in area to half the original figure.

139. A trapezium is such, that the perpendiculars let fall on a diagonal from the opposite angles are equal. Divide the trapezium into four equal triangles, by straight lines drawn to the angles from a point within it.

140. If two opposite sides of a trapezium be parallel to one another, the straight line joining their bisections, bisects the trapezium.

141. If of the four triangles into which the diagonals divide a trapezium, any two opposite ones are equal, the trapezium has two of its opposite sides parallel.

142. If two sides of a quadrilateral are parallel but not equal, and the other two sides are equal but not parallel, the opposite angles of the quadrilateral are together equal to two right angles: and conversely.

143. If two sides of a quadrilateral be parallel, and the line joining the middle points of the diagonals be produced to meet the other sides; the line so produced will be equal to half the sum of the parallel sides, and the line between the points of bisection equal to half their difference.

144. To bisect a trapezium, (1) by a line drawn from one of its angular points: (2) by a line drawn from a given point in one side.

145. To divide a square into four equal portions by lines drawn from any point in one of its sides.

146. It is impossible to divide a quadrilateral figure (except it be a parallelogram) into equal triangles by lines drawn from a point within it to its four corners.

### IX.

147. If the greater of the acute angles of a right-angled triangle, be double the other, the square on the greater side is three times the square on the other.

148. Upon a given straight line construct a right-angled triangle such that the square on the other side may be equal to seven times the square on the given line.

149. If from the vertex of a plane triangle, a perpendicular fall upon the base or the base produced, the difference of the squares on the sides is equal to the difference of the squares on the segments of the base.

150. If from the middle point of one of the sides of a right-angled triangle, a perpendicular be drawn to the hypotenuse, the difference of the squares on the segments into which it is divided, is equal to the square on the other side.

151. If a straight line be drawn from one of the acute angles of a right-angled triangle, bisecting the opposite side, the square upon that line is less than the square upon the hypotenuse by three times the square upon half the line bisected.

152. If the sum of the squares on the three sides of a triangle be equal to eight times the square on the line drawn from the vertex to the point of bisection of the base, then the vertical angle is a right angle.

153. If a line be drawn parallel to the hypotenuse of a right-angled triangle, and each of the acute angles be joined with the points where this line intersects the sides respectively opposite to them, the squares on the joining lines are together equal to the squares on the hypotenuse and on the line drawn parallel to it.

154. Let  $ACB$ ,  $ADB$  be two right-angled triangles having a common hypotenuse  $AB$ , join  $CD$ , and on  $CD$  produced both ways draw perpendiculars  $AE$ ,  $BF$ . Shew that  $CE^2 + CF^2 = DE^2 + DF^2$ .

155. If perpendiculars  $AD$ ,  $BE$ ,  $CF$  drawn from the angles on the opposite sides of a triangle intersect in  $G$ , the squares on  $AB$ ,  $BC$ , and  $CA$ , are together three times the squares on  $AG$ ,  $BG$ , and  $CG$ .

156. If  $ABC$  be a triangle of which the angle  $A$  is a right angle; and  $BE$ ,  $CF$  be drawn bisecting the opposite sides respectively: shew that four times the sum of the squares on  $BE$  and  $CF$  is equal to five times the square on  $BC$ .

157. If  $ABC$  be an isosceles triangle, and  $CD$  be drawn perpendicular to  $AB$ ; the sum of the squares on the three sides is equal to

$$AD^2 + 2 \cdot BD^2 + 3 \cdot CD^2.$$

158. The sum of the squares described upon the sides of a rhombus is equal to the squares described on its diameters.

159. A point is taken within a square, and straight lines drawn from it to the angular points of the square, and perpendicular to the sides; the squares on the first are double the sum of the squares on the last. Shew that these sums are least when the point is in the center of the square.

160. In the figure Euc. I. 47,

(a) Shew that the diagonals  $FA$ ,  $AK$  of the squares on  $AB$ ,  $AC$ , lie in the same straight line.

(b) If  $DF$ ,  $EK$  be joined, the sum of the angles at the bases of the triangles  $BFD$ ,  $CEK$  is equal to one right angle.

(c) If  $BG$  and  $CH$  be joined, those lines will be parallel.

(d) If perpendiculars be let fall from  $F$  and  $K$  on  $BC$  produced, the parts produced will be equal; and the perpendiculars together will be equal to  $BC$ .

(e) Join  $GH$ ,  $KE$ ,  $FD$ , and prove that each of the triangles so formed, equals the given triangle  $ABC$ .

(f) The sum of the squares on  $GH$ ,  $KE$ , and  $FD$  will be equal to six times the square on the hypotenuse.

(g) The difference of the squares on  $AB$ ,  $AC$ , is equal to the difference of the squares on  $AD$ ,  $AE$ .

161. The area of any two parallelograms described on the two sides of a triangle, is equal to that of a parallelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle, to the intersection of the two sides of the former parallelograms produced to meet.

162. If one angle of a triangle be a right angle, and another equal to two-thirds of a right angle, prove from the First Book of Euclid, that the equilateral triangle described on the hypotenuse, is equal to the sum of the equilateral triangles described upon the sides which contain the right angle.

## BOOK II.

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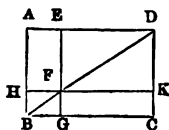
### DEFINITIONS.

#### I.

EVERY right-angled parallelogram is called a *rectangle*, and is said to be contained by any two of the straight lines which contain one of the right angles.

#### II.

In every parallelogram, any of the parallelograms about a diameter together with the two complements, is called a *gnomon*.



"Thus the parallelogram *HG* together with the complements *AF*, *FC*, is the gnomon, which is more briefly expressed by the letters *AGK*, or *EHC*, which are at the opposite angles of the parallelograms which make the gnomon."

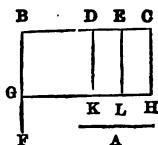
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### PROPOSITION I. THEOREM.

*If there be two straight lines, one of which is divided into any number of parts; the rectangle contained by the two straight lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.*

Let *A* and *BC* be two straight lines;  
and let *BC* be divided into any parts *BD*, *DE*, *EC*, in the points *D*, *E*.

Then the rectangle contained by the straight lines *A* and *BC*, shall be equal to the rectangle contained by *A* and *BD*, together with that contained by *A* and *DE*, and that contained by *A* and *EC*.

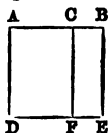


From the point  $B$ , draw  $BF$  at right angles to  $BC$ , (I. 11.)  
 and make  $BG$  equal to  $A$ ; (I. 3.)  
 through  $G$  draw  $GH$  parallel to  $BC$ , (I. 31.)  
 and through  $D, E, C$ , draw  $DK, EL, CH$  parallel to  $BG$ , meeting  
 $GH$  in  $K, L, H$ .  
 Then the rectangle  $BH$  is equal to the rectangles  $BK, DL, EH$ .  
 And  $BH$  is contained by  $A$  and  $BC$ ,  
 for it is contained by  $GB, BC$ , and  $GB$  is equal to  $A$ :  
 and the rectangle  $BK$  is contained by  $A, BD$ ,  
 for it is contained by  $GB, BD$ , of which  $GB$  is equal to  $A$ :  
 also  $DL$  is contained by  $A, DE$ ,  
 because  $DK$ , that is,  $BG$ , (I. 34.) is equal to  $A$ ;  
 and in like manner the rectangle  $EH$  is contained by  $A, EC$ :  
 therefore the rectangle contained by  $A, BC$ , is equal to the several  
 rectangles contained by  $A, BD$ , and by  $A, DE$ , and by  $A, EC$ .  
 Wherefore, if there be two straight lines, &c. Q. E. D.

## PROPOSITION II. THEOREM.

*If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square on the whole line.*

Let the straight line  $AB$  be divided into any two parts in the point  $C$ .  
 Then the rectangle contained by  $AB, BC$ , together with that contained by  $AB, AC$ , shall be equal to the square on  $AB$ .



Upon  $AB$  describe the square  $ADEB$ , (I. 46.) and through  $C$  draw  
 $CF$  parallel to  $AD$  or  $BE$ , (I. 31.) meeting  $DE$  in  $F$ .

Then  $AE$  is equal to the rectangles  $AF, CE$ .

And  $AE$  is the square on  $AB$ ;

and  $AF$  is the rectangle contained by  $BA, AC$ ;

for it is contained by  $DA, AC$ , of which  $DA$  is equal to  $AB$ :

and  $CE$  is contained by  $AB, BC$ ,

for  $BE$  is equal to  $AB$ :

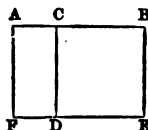
therefore the rectangle contained by  $AB, AC$ , together with the  
 rectangle  $AB, BC$  is equal to the square on  $AB$ .

If therefore a straight line, &c. Q. E. D.

PROPOSITION III. THEOREM.

*If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square on the aforesaid part.*

Let the straight line  $AB$  be divided into any two parts in the point  $C$ .  
Then the rectangle  $AB, BC$ , shall be equal to the rectangle  $AC, CB$ , together with the square on  $BC$ .



Upon  $BC$  describe the square  $CDEB$ , (I. 46.) and produce  $ED$  to  $F$ , through  $A$  draw  $AF$  parallel to  $CD$  or  $BE$ , (I. 31.) meeting  $EF$  in  $F$ .

Then the rectangle  $AE$  is equal to the rectangles  $AD, CE$ .

And  $AE$  is the rectangle contained by  $AB, BC$ ,

for it is contained by  $AB, BE$ , of which  $BE$  is equal to  $BC$ :

and  $AD$  is contained by  $AC, CB$ , for  $CD$  is equal to  $CB$ :

and  $CE$  is the square on  $BC$ :

therefore the rectangle  $AB, BC$ , is equal to the rectangle  $AC, CB$ , together with the square on  $BC$ .

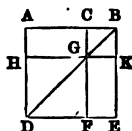
If therefore a straight line be divided, &c. Q.E.D.

PROPOSITION IV. THEOREM.

*If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the parts.*

Let the straight line  $AB$  be divided into any two parts in  $C$ .

Then the square on  $AB$  shall be equal to the squares on  $AC$ , and  $CB$ , together with twice the rectangle contained by  $AC, CB$ .



Upon  $AB$  describe the square  $ADEB$ , (I. 46.) join  $BD$ , through  $C$  draw  $CGF$  parallel to  $AD$  or  $BE$ , (I. 31.) meeting  $BD$  in  $G$  and  $DE$  in  $F$ ;

and through  $G$  draw  $HGK$  parallel to  $AB$  or  $DE$ , meeting  $AD$  in  $H$ , and  $BE$  in  $K$ ;

Then, because  $CF$  is parallel to  $AD$  and  $BD$  falls upon them, therefore the exterior angle  $BGC$  is equal to the interior and opposite angle  $BDA$ ; (I. 29.)

but the angle  $BDA$  is equal to the angle  $DBA$ , (I. 5.)

because  $BA$  is equal to  $AD$ , being sides of a square;



wherefore the angle  $BGC$  is equal to the angle  $DBA$  or  $GBC$ ;  
 and therefore the side  $BC$  is equal to the side  $CG$ ; (I. 6.)  
 but  $BC$  is equal also to  $GK$ , and  $CG$  to  $BK$ ; (I. 34.)  
 wherefore the figure  $CGKB$  is equilateral.

It is likewise rectangular;  
 for, since  $CG$  is parallel to  $BK$ , and  $BC$  meets them,  
 therefore the angles  $KBC$ ,  $BCG$  are equal to two right angles; (I. 29.)  
 but the angle  $KBC$  is a right angle; (def. 30. constr.)  
 wherefore  $BCG$  is a right angle:

and therefore also the angles  $CGK$ ,  $GKB$ , opposite to these, are right angles; (I. 34.)

wherefore  $CGKB$  is rectangular:

but it is also equilateral, as was demonstrated;  
 wherefore it is a square, and it is upon the side  $CB$ .

For the same reason  $HF$  is a square,  
 and it is upon the side  $HG$ , which is equal to  $AC$ . (I. 34.)

Therefore the figures  $HF$ ,  $CK$ , are the squares on  $AC$ ,  $CB$ .

And because the complement  $AG$  is equal to the complement  $GE$ ,  
 (I. 43.)

and that  $AG$  is the rectangle contained by  $AC$ ,  $CB$ ,  
 for  $GC$  is equal to  $CB$ ;

therefore  $GE$  is also equal to the rectangle  $AC$ ,  $CB$ ;

wherefore  $AG$ ,  $GE$  are equal to twice the rectangle  $AC$ ,  $CB$ ;

and  $HF$ ,  $CK$  are the squares on  $AC$ ,  $CB$ ;

wherefore the four figures  $HF$ ,  $CK$ ,  $AG$ ,  $GE$ , are equal to the  
 squares on  $AC$ ,  $CB$ , and twice the rectangle  $AC$ ,  $CB$ :

but  $HF$ ,  $CK$ ,  $AG$ ,  $GE$  make up the whole figure  $ADEB$ , which  
 is the square on  $AB$ ;

therefore the square on  $AB$  is equal to the squares on  $AC$ ,  $CB$ , and  
 twice the rectangle  $AC$ ,  $CB$ .

Wherefore, if a straight line be divided, &c. Q.E.D.

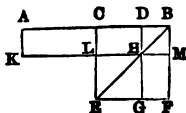
COR. From the demonstration, it is manifest, that the parallelograms about the diameter of a square, are likewise squares.

### PROPOSITION V. THEOREM.

*If a straight line be divided into two equal parts, and also into two unequal parts; the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.*

Let the straight line  $AB$  be divided into two equal parts in the point  $C$ , and into two unequal parts in the point  $D$ .

Then the rectangle  $AD$ ,  $DB$ , together with the square on  $CD$ , shall be equal to the square on  $CB$ .



Upon  $CB$  describe the square  $CEFB$ , (I. 46.) join  $BE$ ,  
 through  $D$  draw  $DHG$  parallel to  $CE$  or  $BF$ , (I. 31.) meeting  $DE$   
 in  $H$ , and  $EF$  in  $G$ ,  
 and through  $H$  draw  $KLM$  parallel to  $CB$  or  $EF$ , meeting  $CE$   
 in  $L$ , and  $BF$  in  $M$ ;  
 also through  $A$  draw  $AK$  parallel to  $CL$  or  $BM$ , meeting  $MLK$  in  $K$ .  
 Then because the complement  $CH$  is equal to the complement  $HF$ ,  
 (I. 43.) to each of these equals add  $DM$ ;  
     therefore the whole  $CM$  is equal to the whole  $DF$ ;  
     but because the line  $AC$  is equal to  $CB$ ,  
     therefore  $AL$  is equal to  $CM$ , (I. 36.)  
     therefore also  $AL$  is equal to  $DF$ ;  
     to each of these equals add  $CH$ ,  
     and therefore the whole  $AH$  is equal to  $DF$  and  $CH$ ;  
 but  $AH$  is the rectangle contained by  $AD$ ,  $DB$ , for  $DH$  is equal to  $DB$ ;  
     and  $DF$  together with  $CH$  is the gnomon  $CMG$ ;  
 therefore the gnomon  $CMG$  is equal to the rectangle  $AD$ ,  $DB$ ;  
 to each of these equals add  $LG$ , which is equal to the square on  
 $CD$ ; (II. 4. Cor.)  
 therefore the gnomon  $CMG$ , together with  $LG$ , is equal to the  
 rectangle  $AD$ ,  $DB$ , together with the square on  $CD$ ;  
 but the gnomon  $CMG$  and  $LG$  make up the whole figure  $CEFB$ ,  
 which is the square on  $CB$ ;  
 therefore the rectangle  $AD$ ,  $DB$ , together with the square on  $CD$   
 is equal to the square on  $CB$ .

Wherefore, if a straight line, &c. Q.E.D.

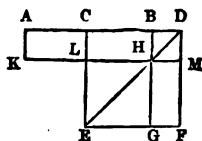
COR. From this proposition it is manifest, that the difference of  
 the squares on two unequal lines  $AC$ ,  $CD$ , is equal to the rectangle  
 contained by their sum  $AD$  and their difference  $DB$ .

#### PROPOSITION VI. THEOREM.

*If a straight line be bisected, and produced to any point; the rectangle  
 contained by the whole line thus produced, and the part of it produced,  
 together with the square on half the line bisected, is equal to the square on  
 the straight line which is made up of the half and the part produced.*

Let the straight line  $AB$  be bisected in  $C$ , and produced to the point  $D$ .

Then the rectangle  $AD$ ,  $DB$ , together with the square on  $CB$ , shall  
 be equal to the square on  $CD$ .



Upon  $CD$  describe the square  $CEFD$ , (I. 46.) and join  $DE$ ,  
 through  $B$  draw  $BHG$  parallel to  $CE$  or  $DF$ , (I. 31.) meeting  $DE$   
 in  $H$ , and  $EF$  in  $G$ ;  
 through  $H$  draw  $KLM$  parallel to  $AD$  or  $EF$ , meeting  $DF$  in  
 $M$ , and  $CE$  in  $L$ ;  
 and through  $A$  draw  $AK$  parallel to  $CL$  or  $DM$ , meeting  $MLK$  in  $K$ .

Then because the line  $AC$  is equal to  $CB$ ,  
therefore the rectangle  $AL$  is equal to the rectangle  $CH$ , (I. 36.)

but  $CH$  is equal to  $HF$ ; (I. 43.)

therefore  $AL$  is equal to  $HF$ ;

to each of these equals add  $CM$ ;

therefore the whole  $AM$  is equal to the gnomon  $CMG$ ;

but  $AM$  is the rectangle contained by  $AD$ ,  $DB$ ,

for  $DM$  is equal to  $DB$ : (II. 4. Cor.)

therefore the gnomon  $CMG$  is equal to the rectangle  $AD$ ,  $DB$ ;

to each of these equals add  $LG$  which is equal to the square on  $CB$ ;

therefore the rectangle  $AD$ ,  $DB$ , together with the square on  $CB$ , is

equal to the gnomon  $CMG$ , and the figure  $LG$ ;

but the gnomon  $CMG$  and  $LG$  make up the whole figure  $CEFD$ ,

which is the square on  $CD$ ;

therefore the rectangle  $AD$ ,  $DB$ , together with the square on  $CB$ ,

is equal to the square on  $CD$ .

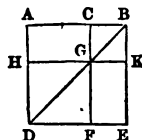
Wherefore, if a straight line, &c. Q. E. D.

#### PROPOSITION VII. THEOREM.

*If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.*

Let the straight line  $AB$  be divided into any two parts in the point  $C$ .

Then the squares on  $AB$ ,  $BC$  shall be equal to twice the rectangle  $AB$ ,  $BC$ , together with the square on  $AC$ .



Upon  $AB$  describe the square  $ADEB$ , (I. 46.) and join  $BD$ ;  
through  $C$  draw  $CF$  parallel to  $AD$  or  $BE$  (I. 31.) meeting  $BD$  in  
 $G$ , and  $DE$  in  $F$ ;

through  $G$  draw  $HGK$  parallel to  $AB$  or  $DE$ , meeting  $AD$  in  $H$ ,  
and  $BE$  in  $K$ .

Then because  $AG$  is equal to  $GE$ , (I. 43.)

add to each of them  $CK$ ;

therefore the whole  $AK$  is equal to the whole  $CE$ ;

and therefore  $AK$ ,  $CE$ , are double of  $AK$ ;

but  $AK$ ,  $CE$ , are the gnomon  $AKF$  and the square  $CK$ ;

therefore the gnomon  $AKF$  and the square  $CK$  are double of  $AK$ ;

but twice the rectangle  $AB$ ,  $BC$ , is double of  $AK$ ,

for  $BK$  is equal to  $BC$ ; (II. 4. Cor.)

therefore the gnomon  $AKF$  and the square  $CK$ , are equal to twice the  
rectangle  $AB$ ,  $BC$ ;

to each of these equals add  $HF$ , which is equal to the square on  $AC$ ,

therefore the gnomon  $AKF$ , and the squares  $CK$ ,  $HF$ , are equal to

twice the rectangle  $AB$ ,  $BC$ , and the square on  $AC$ ;

but the gnomon  $AKF$ , together with the squares  $CK$ ,  $HF$ , make

up the whole figure  $ADEB$  and  $CK$ , which are the squares on  $AB$  and  $BC$ ;

therefore the squares on  $AB$  and  $BC$  are equal to twice the rectangle  $AB, BC$ , together with the square on  $AC$ .

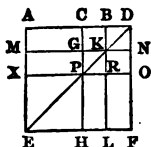
Wherefore, if a straight line, &c. Q. E. D.

**PROPOSITION VIII. THEOREM.**

*If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square on the other part, is equal to the square on the straight line, which is made up of the whole and that part.*

Let the straight line  $AB$  be divided into any two parts in the point  $C$ .

Then four times the rectangle  $AB, BC$ , together with the square on  $AC$ , shall be equal to the square on the straight line made up of  $AB$  and  $BC$  together.



Produce  $AB$  to  $D$ , so that  $BD$  be equal to  $CB$ , (I. 3.)

upon  $AD$  describe the square  $AEDF$ , (I. 46.), and join  $DE$ ,  
through  $B, C$ , draw  $BL, CH$  parallel to  $AE$  or  $DF$ , and cutting  $DE$   
in the points  $K, P$  respectively, and meeting  $EF$  in  $L, H$ ;  
through  $K, P$ , draw  $MGKN, XPRO$  parallel to  $AD$  or  $EF$ .

Then because  $CB$  is equal to  $BD$ ,  $CB$  to  $GK$ , and  $BD$  to  $KN$ ;

therefore  $GK$  is equal to  $KN$ ;

for the same reason,  $PR$  is equal to  $RO$ ;

and because  $CB$  is equal to  $BD$ , and  $GK$  to  $KN$ ,

therefore the rectangle  $CK$  is equal to  $BN$ , and  $GR$  to  $RN$ ; (I. 36.)

but  $CK$  is equal to  $RN$ , (I. 43.)

because they are the complements of the parallelogram  $CO$ ;

therefore also  $BN$  is equal to  $GR$ ;

and the four rectangles  $BN$ ,  $CK$ ,  $GR$ ,  $RN$ , are equal to one another, and so are quadruple of one of them  $CK$ .

Again, because  $CB$  is equal to  $BD$ , and  $BD$  to  $BK$ , that is, to  $CG$ :

and because  $CB$  is equal to  $GK$ , that is, to  $GP$ ;

therefore  $CG$  is equal to  $GP$ .

And because  $CG$  is equal to  $GP$ , and  $PR$  to  $RO$ ,

therefore the rectangle  $AG$  is equal to  $MP$ , and  $PL$  to  $RF$ ;

but the rectangle  $MP$  is equal to  $PL$ , (I. 43.)

because they are the complements of the parallelogram  $ML$ :

wherefore also  $AG$  is equal to  $RF$ :

therefore the four rectangles  $AG$ ,  $MP$ ,  $PL$ ,  $RF$ , are equal to one another, and so are quadruple of one of them  $AG$ .

And it was demonstrated, that the four  $CK, BN, GR$ , and  $RN$ , are quadruple of  $CK$ :

therefore the eight rectangles which contain the gnomon  $AOH$ , are quadruple of  $AK$ .

And because  $AK$  is the rectangle contained by  $AB, BC$ ,  
for  $BK$  is equal to  $BC$ ;

therefore four times the rectangle  $AB, BC$  is quadruple of  $AK$ ;

but the gnomon  $AOH$  was demonstrated to be quadruple of  $AK$ ;  
therefore four times the rectangle  $AB, BC$  is equal to the gnomon  $AOH$ ;  
to each of these equals add  $XH$ , which is equal to the square on  $AC$ ;  
therefore four times the rectangle  $AB, BC$ , together with the square  
on  $AC$ , is equal to the gnomon  $AOH$  and the square  $XH$ ;  
but the gnomon  $AOH$  and  $XH$  make up the figure  $AEFD$ , which is  
the square on  $AD$ ;

therefore four times the rectangle  $AB, BC$  together with the square  
on  $AC$ , is equal to the square on  $AD$ , that is, on  $AB$  and  $BC$  added  
together in one straight line.

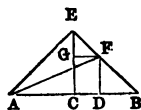
Wherefore, if a straight line, &c, Q.E.D.

### PROPOSITION IX. THEOREM.

*If a straight line be divided into two equal, and also into two unequal parts; the squares on the two unequal parts are together double of the square on half the line, and of the square on the line between the points of section.*

Let the straight line  $AB$  be divided into two equal parts in the point  $C$ , and into two unequal parts in the point  $D$ .

Then the squares on  $AD, DB$  together, shall be double of the squares on  $AC, CD$ .



From the point  $C$  draw  $CE$  at right angles to  $AB$ , (I. 11.)

make  $CE$  equal to  $AC$  or  $CB$ , (I. 3.) and join  $EA, EB$ ;

through  $D$  draw  $DF$  parallel to  $CE$ , meeting  $EB$  in  $F$ , (I. 31.)

through  $F$  draw  $FG$  parallel to  $BA$ , and join  $AF$ .

Then, because  $AC$  is equal to  $CE$ ,

therefore the angle  $AEC$  is equal to the angle  $EAC$ ; (I. 5.)

and because  $ACE$  is a right angle,

therefore the two other angles  $AEC, EAC$  of the triangle are together  
equal to a right angle; (I. 32.)

and since they are equal to one another;

therefore each of them is half a right angle.

For the same reason, each of the angles  $CEB, EBC$  is half a right angle;

and therefore the whole  $AEB$  is a right angle.

And because the angle  $GEF$  is half a right angle,

and  $EGF$  a right angle,

for it is equal to the interior and opposite angle  $ECB$ , (I. 29.)

therefore the remaining angle  $EBF$  is half a right angle;

wherefore the angle  $GEF$  is equal to the angle  $EBF$ ,

and the side  $GF$  equal to the side  $EG$ . (I. 6.)

Again, because the angle at  $B$  is half a right angle,  
and  $FDB$  a right angle,  
for it is equal to the interior and opposite angle  $ECB$ , (I. 29.)  
therefore the remaining angle  $BFD$  is half a right angle;  
wherefore the angle at  $B$  is equal to the angle  $BFD$ ,  
and the side  $DF$  equal to the side  $DB$ . (I. 6.)

And because  $AC$  is equal to  $CE$ ,  
the square on  $AC$  is equal to the square on  $CE$ ;  
therefore the squares on  $AC$ ,  $CE$  are double of the square on  $AC$ ;  
but the square on  $AE$  is equal to the squares on  $AC$ ,  $CE$ , (I. 47).

because  $\angle ACE$  is a right angle;  
therefore the square on  $AE$  is double of the square on  $AC$ .

Again, because  $EG$  is equal to  $GF$ ,  
the square on  $EG$  is equal to the square on  $GF$ ;  
therefore the squares on  $EG$ ,  $GF$  are double of the square on  $GF$ ;  
but the square on  $EF$  is equal to the squares on  $EG$ ,  $GF$ ; (I. 47).

therefore the square on  $EF$  is double of the square on  $GF$ ;  
and  $GF$  is equal to  $CD$ ; (I. 34.)

therefore the square on  $EF$  is double of the square on  $CD$ ;  
but the square on  $AE$  is double of the square on  $AC$ ;

but the square on  $AE$  is double of the square on  $EC$ ,  
 fore the squares on  $AE, EF$  are double of the squares on  $AC$ ,  
 but the square on  $AF$  is equal to the squares on  $AE, EF$ .

therefore the square on  $AF$  is double of the squares on  $AC$ ,  $CD$ :

but the squares on  $AD$ ,  $DF$  are equal to the square on  $AF$ ;  
because the angle  $ADF$  is a right angle; (I. 47.)

therefore the squares on  $AD$ ,  $DF$  are double of the squares on  $AC$ ,  $CD$ ;  
and  $DF$  is equal to  $DB$ ;

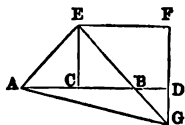
therefore the squares on  $AD, DB$  are double of the squares on  $AC, CD$ .  
If therefore a straight line be divided, &c. Q.E.D.

**PROPOSITION X. THEOREM.**

*If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected, and of the square on the line made up of the half and the part produced.*

Let the straight line  $AB$  be bisected in  $C$ , and produced to the point  $D$ .

Then the squares on  $AD$ ,  $DB$ , shall be double of the squares on  $AC$ ,  $CD$ .



From the point  $C$  draw  $CE$  at right angles to  $AB$ , (I. 11.)

make  $CE$  equal to  $AC$  or  $CB$ , (I. 3.) and join  $AE$ ,  $EB$ ;

through  $E$  draw  $EF$  parallel to  $AB$ , (I. 31.)

and through  $D$  draw  $DF$  parallel to  $CE$ , meeting  $EF$  in  $F$ .

Then because the straight line  $EF$  meets the parallels  $CE$ ,  $FD$ ,  
therefore the angles  $CEF$ ,  $EFD$  are equal to two right angles; (I. 29.)  
and therefore the angles  $BEF$ ,  $EFD$  are less than two right angles.

But straight lines, which with another straight line make the interior angles upon the same side of a line, less than two right angles, will meet if produced far enough; (I. ax. 12.)

therefore  $EB$ ,  $FD$  will meet, if produced towards  $B$ ,  $D$ ;

let them be produced and meet in  $G$ , and join  $AG$ .

Then, because  $AC$  is equal to  $CE$ ,

therefore the angle  $CEA$  is equal to the angle  $EAC$ ; (I. 5.)

and the angle  $ACE$  is a right angle;

therefore each of the angles  $CEA$ ,  $EAC$  is half a right angle. (I. 32.)

For the same reason,

each of the angles  $CEB$ ,  $EBC$  is half a right angle;

therefore the whole  $AEB$  is a right angle.

And because  $EBC$  is half a right angle,

therefore  $DBG$  is also half a right angle, (I. 15.)

for they are vertically opposite;

but  $BDG$  is a right angle,

because it is equal to the alternate angle  $DCE$ ; (I. 29.)

therefore the remaining angle  $DGB$  is half a right angle;

and is therefore equal to the angle  $DBG$ ;

wherefore also the side  $BD$  is equal to the side  $DG$ . (I. 6.)

Again, because  $EGF$  is half a right angle, and the angle at  $F$  is a right angle, being equal to the opposite angle  $ECD$ , (I. 34.)

therefore the remaining angle  $FEG$  is half a right angle,

and therefore equal to the angle  $EGF$ ;

wherefore also the side  $GF$  is equal to the side  $FE$ . (I. 6.)

And because  $EC$  is equal to  $CA$ ;

the square on  $EC$  is equal to the square on  $CA$ ;

therefore the squares on  $EC$ ,  $CA$  are double of the square on  $CA$ ;

but the square on  $EA$  is equal to the squares on  $EC$ ,  $CA$ ; (I. 47.)

therefore the square on  $EA$  is double of the square on  $AC$ .

Again, because  $GF$  is equal to  $FE$ ,

the square on  $GF$  is equal to the square on  $FE$ ;

therefore the squares on  $GF$ ,  $FE$  are double of the square on  $FE$ ;

but the square on  $EG$  is equal to the squares on  $GF$ ,  $FE$ ; (I. 47.)

therefore the square on  $EG$  is double of the square on  $FE$ ;

and  $FE$  is equal to  $CD$ ; (I. 34.)

wherefore the square on  $EG$  is double of the square on  $CD$ ;

but it was demonstrated,

that the square on  $EA$  is double of the square on  $AC$ ;

therefore the squares on  $EA$ ,  $EG$  are double of the squares on  $AC$ ,  $CD$ ;

but the square on  $AG$  is equal to the squares on  $EA$ ,  $EG$ ; (I. 47.)

therefore the square on  $AG$  is double of the squares on  $AC$ ,  $CD$ ;

but the squares on  $AD$ ,  $DG$  are equal to the square on  $AG$ ;

therefore the squares on  $AD$ ,  $DG$  are double of the squares on  $AC$ ,  $CD$ ;

but  $DG$  is equal to  $DB$ ;

therefore the squares on  $AD$ ,  $DB$  are double of the squares on  $AC$ ,  $CD$ .

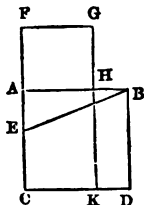
Wherefore, if a straight line, &c. Q.E.D.

## PROPOSITION XI. PROBLEM.

*To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square on the other part.*

Let  $AB$  be the given straight line.

It is required to divide  $AB$  into two parts, so that the rectangle contained by the whole line and one of the parts, shall be equal to the square on the other part.



Upon  $AB$  describe the square  $ACDB$ ; (I. 46.)

bisect  $AC$  in  $E$ , (I. 10.) and join  $BE$ ;

produce  $CA$  to  $F$ , and make  $EF$  equal to  $EB$ , (I. 3.)

upon  $AF$  describe the square  $FGHA$ . (I. 46.)

Then  $AB$  shall be divided in  $H$ , so that the rectangle  $AB, BH$  is equal to the square on  $AH$ .

Produce  $GH$  to meet  $CD$  in  $K$ .

Then because the straight line  $AC$  is bisected in  $E$ , and produced to  $F$ , therefore the rectangle  $CF, FA$  together with the square on  $AE$ , is equal to the square on  $EF$ ; (II. 6.)

but  $EF$  is equal to  $EB$ ;

therefore the rectangle  $CF, FA$  together with the square on  $AE$ , is equal to the square on  $EB$ ;

but the squares on  $BA, AE$  are equal to the square on  $EB$ , (I. 47.)

because the angle  $EAB$  is a right angle;

therefore the rectangle  $CF, FA$ , together with the square on  $AE$ , is equal to the squares on  $BA, AE$ ;

take away the square on  $AE$ , which is common to both;

therefore the rectangle contained by  $CF, FA$  is equal to the square on  $BA$ .

But the figure  $FK$  is the rectangle contained by  $CF, FA$ ,

for  $FA$  is equal to  $FG$ ;

and  $AD$  is the square on  $AB$ ;

therefore the figure  $FK$  is equal to  $AD$ ;

take away the common part  $AK$ ,

therefore the remainder  $FH$  is equal to the remainder  $HD$ ;

but  $HD$  is the rectangle contained by  $AB, BH$ ,

for  $AB$  is equal to  $BD$ ;

and  $FH$  is the square on  $AH$ ;

therefore the rectangle  $AB, BH$ , is equal to the square on  $AH$ .

Wherefore the straight line  $AB$  is divided in  $H$ , so that the rectangle  $AB, BH$  is equal to the square on  $AH$ . Q.E.F.

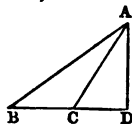


## PROPOSITION XII. THEOREM.

*In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle, is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.*

Let  $ABC$  be an obtuse-angled triangle, having the obtuse angle  $ACB$ , and from the point  $A$ , let  $AD$  be drawn perpendicular to  $BC$  produced.

Then the square on  $AB$  shall be greater than the squares on  $AC$ ,  $CB$ , by twice the rectangle  $BC$ ,  $CD$ .



Because the straight line  $BD$  is divided into two parts in the point  $C$ , therefore the square on  $BD$  is equal to the squares on  $BC$ ,  $CD$ , and twice the rectangle  $BC$ ,  $CD$ ; (II. 4.)

to each of these equals add the square on  $DA$ ;

therefore the squares on  $BD$ ,  $DA$  are equal to the squares on  $BC$ ,  $CD$ ,  $DA$ , and twice the rectangle  $BC$ ,  $CD$ ;

but the square on  $BA$  is equal to the squares on  $BD$ ,  $DA$ , (I. 47.)

because the angle at  $D$  is a right angle;

and the square on  $CA$  is equal to the squares on  $CD$ ,  $DA$ ;

therefore the square on  $BA$  is equal to the squares on  $BC$ ,  $CA$ , and twice the rectangle  $BC$ ,  $CD$ ;

that is, the square on  $BA$  is greater than the squares on  $BC$ ,  $CA$ , by twice the rectangle  $BC$ ,  $CD$ .

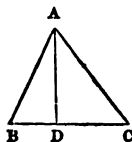
Therefore in obtuse-angled triangles, &c. Q.E.D.

## PROPOSITION XIII. THEOREM.

*In every triangle, the square on the side subtending either of the acute angles, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular let fall upon it from the opposite angle.*

Let  $ABC$  be any triangle, and the angle at  $B$  one of its acute angles, and upon  $BC$ , one of the sides containing it, let fall the perpendicular  $AD$  from the opposite angle. (I. 12.)

Then the square on  $AC$  opposite to the angle  $B$ , shall be less than the squares on  $CB$ ,  $BA$ , by twice the rectangle  $CB$ ,  $BD$ .



First, let  $AD$  fall within the triangle  $ABC$ .

Then because the straight line  $CB$  is divided into two parts in  $D$ , the squares on  $CB$ ,  $BD$  are equal to twice the rectangle contained by  $CB$ ,  $BD$ , and the square on  $DC$ ; (II. 7.)

to each of these equals add the square on  $AD$ ;

therefore the squares on  $CB$ ,  $BD$ ,  $DA$ , are equal to twice the rectangle  $CB$ ,  $BD$ , and the squares on  $AD$ ,  $DC$ ;

but the square on  $AB$  is equal to the squares on  $BD$ ,  $DA$ , (I. 47.)

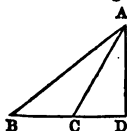
because the angle  $BDA$  is a right angle;

and the square on  $AC$  is equal to the squares on  $AD$ ,  $DC$ ;

therefore the squares on  $CB$ ,  $BA$  are equal to the square on  $AC$ , and twice the rectangle  $CB$ ,  $BD$ ;

that is, the square on  $AC$  alone is less than the squares on  $CB$ ,  $BA$ , by twice the rectangle  $CB$ ,  $BD$ .

Secondly, let  $AD$  fall without the triangle  $ABC$ .



Then, because the angle at  $D$  is a right angle,

the angle  $ACB$  is greater than a right angle; (I. 16.)

and therefore the square on  $AB$  is equal to the squares on  $AC$ ,  $CB$ , and twice the rectangle  $BC$ ,  $CD$ ; (II. 12.)

to each of these equals add the square on  $BC$ ;

therefore the squares on  $AB$ ,  $BC$  are equal to the square on  $AC$ , twice the square on  $BC$ , and twice the rectangle  $BC$ ,  $CD$ ;

but because  $BD$  is divided into two parts in  $C$ ,

therefore the rectangle  $DB$ ,  $BC$  is equal to the rectangle  $BC$ ,  $CD$ , and the square on  $BC$ ; (II. 3.)

and the doubles of these are equal;

that is, twice the rectangle  $DB$ ,  $BC$  is equal to twice the rectangle  $BC$ ,  $CD$  and twice the square on  $BC$ ;

therefore the squares on  $AB$ ,  $BC$  are equal to the square on  $AC$ ,

and twice the rectangle  $DB$ ,  $BC$ ;

wherefore the square on  $AC$  alone is less than the squares on  $AB$ ,  $BC$ , by twice the rectangle  $DB$ ,  $BC$ .

Lastly, let the side  $AC$  be perpendicular to  $BC$ .



Then  $BC$  is the straight line between the perpendicular and the acute angle at  $B$ ;

and it is manifest, that the squares on  $AB$ ,  $BC$ , are equal to the square on  $AC$ , and twice the square on  $BC$ . (I. 47.)

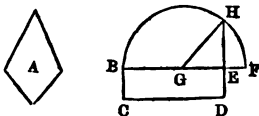
Therefore in any triangle, &c. Q. E. D.

## PROPOSITION XIV. PROBLEM.

To describe a square that shall be equal to a given rectilinear figure.

Let  $A$  be the given rectilinear figure.

It is required to describe a square that shall be equal to  $A$ .



Describe the rectangular parallelogram  $BCDE$  equal to the rectilinear figure  $A$ . (I. 45.)

Then, if the sides of it,  $BE$ ,  $ED$ , are equal to one another, it is a square, and what was required is now done.

But if  $BE$ ,  $ED$ , are not equal, produce one of them  $BE$  to  $F$ , and make  $EF$  equal to  $ED$ , bisect  $BF$  in  $G$ ; (I. 10.)

from the center  $G$ , at the distance  $GB$ , or  $GF$ , describe the semicircle  $BHF$ ,

and produce  $DE$  to meet the circumference in  $H$ .

The square described upon  $EH$  shall be equal to the given rectilinear figure  $A$ .

Join  $GH$ .

Then because the straight line  $BF$  is divided into two equal parts in the point  $G$ , and into two unequal parts in the point  $E$ ; therefore the rectangle  $BE$ ,  $EF$ , together with the square on  $EG$ , is equal to the square on  $GF$ ; (II. 5.)

but  $GF$  is equal to  $GH$ ; (def. 15.)

therefore the rectangle  $BE$ ,  $EF$ , together with the square on  $EG$ , is equal to the square on  $GH$ ;

but the squares on  $HE$ ,  $EG$  are equal to the square on  $GH$ ; (I. 47.) therefore the rectangle  $BE$ ,  $EF$ , together with the square on  $EG$ , is equal to the squares on  $HE$ ,  $EG$ ;

take away the square on  $EG$ , which is common to both;

therefore the rectangle  $BE$ ,  $EF$  is equal to the square on  $HE$ .

But the rectangle contained by  $BE$ ,  $EF$  is the parallelogram  $BD$ , because  $EF$  is equal to  $ED$ ;

therefore  $BD$  is equal to the square on  $EH$ ;

but  $BD$  is equal to the rectilinear figure  $A$ ; (constr.)

therefore the square on  $EH$  is equal to the rectilinear figure  $A$ .

Wherefore a square has been made equal to the given rectilinear figure  $A$ , namely, the square described upon  $EH$ . Q. E. F.

## NOTES TO BOOK II.

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IN Book I, Geometrical magnitudes of the same kind, lines, angles and surfaces, more particularly triangles and parallelograms, are compared, either as being absolutely equal, or unequal to one another.

IN Book II, the properties of right-angled parallelograms, but without reference to their magnitudes, are demonstrated, and an important extension is made of Euc. I. 47, to acute-angled and obtuse-angled triangles. Euclid has given no definition of a *rectangular parallelogram* or *rectangle*: probably, because the Greek expression *παλληλόγραμμον ὀρθογώνιον*, or *ὀρθογώνιον* simply, is a definition of the figure. In English, the term *rectangle*, formed from *rectus* *angulus*, ought to be defined before its properties are demonstrated. A rectangle may be defined to be a parallelogram having one angle a right-angle, or a right angled parallelogram; and a square is a rectangle having all its sides equal.

As the squares in Euclid's demonstrations are squares described or supposed to be described on straight lines, the expression "*the square on AB*," is a more appropriate abbreviation for "*the square described on the line AB*," than "*the square of AB*." The latter expression more fitly expresses the arithmetical or algebraical equivalent for the square on the line *AB*.

IN Euc. I. 35, it may be seen that there may be an indefinite number of parallelograms on the same base and between the same parallels whose areas are always equal to one another; but that one of them has all its angles right angles, and the length of its boundary less than the boundary of any other parallelogram upon the same base and between the same parallels. The area of this rectangular parallelogram is therefore determined by the two lines which contain one of its right angles. Hence it is stated in Def. 1, that every right-angled parallelogram *is said to be contained* by any two of the straight lines which contain one of the right angles. No distinction is made in Book II, between *equality* and *identity*, as the rectangle may be said to be contained by two lines which are equal respectively to the two which contain one right angle of the figure. It may be remarked that the rectangle itself *is bounded* by four straight lines.

It is of primary importance to discriminate the Geometrical conception of a rectangle from the Arithmetical or Algebraical representation of it. The subject of Geometry is *magnitude* not *number*, and therefore it would be a departure from strict reasoning on space, to substitute in Geometrical demonstrations, the Arithmetical or Algebraical representation of a rectangle for the rectangle itself. It is however, absolutely necessary that the connexion of *number* and *magnitude* be clearly understood, as far as regards the representation of lines and areas.

All lines are measured by lines, and all surfaces by surfaces. Some one line of definite length is arbitrarily assumed as the linear unit, and the length of every other line is represented by the number of linear units, contained in it. The square is the figure assumed for the measure of surfaces. The square unit or the unit of area is assumed to be that square, the side of which is one unit in length, and the magnitude of every surface is represented by the number of square units contained in it. But here it may be remarked, that the properties of rectangles and squares in the Second Book of Euclid are proved independently

of the consideration, whether the sides of the rectangles can be represented by any multiples of the same linear unit. If, however, the sides of rectangles are supposed to be divisible into an exact number of linear units, a numerical representation for the area of a rectangle may be deduced.

On two lines at right angles to each other, take  $AB$  equal to 4, and  $AD$  equal to 3 linear units.

Complete the rectangle  $ABCD$ , and through the points of division of  $AB$ ,  $AD$ , draw  $EL$ ,  $FM$ ,  $GN$  parallel to  $AD$ ; and  $HP$ ,  $KQ$  parallel to  $AB$  respectively.



Then the whole rectangle  $AC$  is divided into squares, all equal to each other.

And  $AC$  is equal to the sum of the rectangles  $AL$ ,  $EM$ ,  $FN$ ,  $GC$ ; (II. 1.) also these rectangles are equal to one another, (I. 36.)

therefore the whole  $AC$  is equal to four times one of them  $AL$ .

Again, the rectangle  $AL$  is equal to the rectangles  $EH$ ,  $HR$ ,  $RD$ , and these rectangles, by construction, are squares described upon the equal lines  $AH$ ,  $HK$ ,  $KD$ , and are equal to one another.

Therefore the rectangle  $AL$  is equal to 3 times the square on  $AH$ , but the whole rectangle  $AC$  is equal to 4 times the rectangle  $AL$ , therefore the rectangle  $AC$  is 4  $\times$  3 times the square on  $AH$ , or 12 square units:

that is, the product of the two numbers which express the number of linear units in the two sides, will give the number of square units in the rectangle, and therefore will be an arithmetical representation of its area.

And generally, if  $AB$ ,  $AD$ , instead of 4 and 3, consisted of  $a$  and  $b$  linear units respectively, it may be shewn in a similar manner, that the area of the rectangle  $AC$  would contain  $ab$  square units; and therefore the product  $ab$  is a proper representation for the area of the rectangle  $AC$ .

Hence, it follows, that the term *rectangle* in Geometry corresponds to the term *product* in Arithmetic and Algebra, and that a similar comparison may be made between the products of the two numbers which represent the sides of rectangles, as between the areas of the rectangles themselves. This forms the basis of what are called Arithmetical or Algebraical proofs of Geometrical properties.

If the two sides of the rectangle be equal, or if  $b$  be equal to  $a$ , the figure is a square, and the area is represented by  $aa$  or  $a^2$ .

Also, since a triangle is equal to the half of a parallelogram of the same base and altitude;

Therefore the area of a triangle will be represented by half the rectangle which has the same base and altitude as the triangle: in other words, if the length of the base be  $a$  units, and the altitude be  $b$  units;

Then the area of the triangle is algebraically represented by  $\frac{1}{2}ab$ .

The demonstrations of the first eight propositions, exemplify the obvious axiom, that, "the whole area of every figure in each case, is equal to all the parts of it taken together."

Def. 2. The parallelogram  $EK$  together with the complements  $AF$

*FC*, is also a *gnomon*, as well as the parallelogram *HG* together with the same complements.

Prop. i. For the sake of brevity of expression, "the rectangle contained by the straight lines *AB*, *BC*," is called "the rectangle *AB*, *BC*;" and sometimes "the rectangle *ABC*."

To this proposition may be added the corollary: If two straight lines be divided into any number of parts, the rectangle contained by the two straight lines, is equal to the rectangles contained by the several parts of one line and the several parts of the other respectively.

The method of reasoning on the properties of rectangles, by means of the products which indicate the number of square units contained in their areas, is foreign to Euclid's ideas of rectangles, as discussed in his Second Book, which have no reference to any particular unit of length or measure of surface.

Prop. i. The figures *BH*, *BK*, *DL*, *EH* are rectangles, as may readily be shewn. For, by the parallels, the angle *CEL* is equal to *EDK*; and the angle *EDK* is equal to *BDG* (Euc. i. 29.). But *BDG* is a right angle. Hence one of the angles in each of the figures *BH*, *BK*, *DL*, *EH* is a right angle, and therefore (Euc. i. 46, Cor.) these figures are rectangular.

Prop. i. Algebraically. (fig. Prop. i.)

Let the line *BC* contain *a* linear units, and the line *A*, *b* linear units of the same length.

Also suppose the parts *BD*, *DE*, *EC* to contain *m*, *n*, *p* linear units respectively.

$$\text{Then } a = m + n + p,$$

multiply these equals by *b*,

$$\text{therefore } ab = bm + bn + bp.$$

That is, the product of two numbers, one of which is divided into any number of parts, is equal to the sum of the products of the undivided number, and the several parts of the other;

or, if the Geometrical interpretation of the products be restored,

The number of square units expressed by the product *ab*, is equal to the number of square units expressed by the sum of the products *bm*, *bn*, *bp*.

Prop. ii. Algebraically. (fig. Prop. ii.)

Let *AB* contain *a* linear units, and *AC*, *CB*, *m* and *n* linear units respectively.

$$\text{Then } m + n = a,$$

multiply these equals by *a*,

$$\text{therefore } am + an = a^2.$$

That is, if a number be divided into any two parts, the sum of the products of the whole and each of the parts is equal to the square of the whole number

Prop. iii. Algebraically. (fig. Prop. iii.)

Let *AB* contain *a* linear units, and let *BC* contain *m*, and *AC*, *n* linear units.

$$\text{Then } a = m + n,$$

multiply these equals by *m*,

$$\text{therefore } ma = m^2 + mn.$$

That is, if a number be divided into any two parts, the product of the whole number and one of the parts, is equal to the square of that part, and the product of the two parts.

Prop. iv. might have been deduced from the two preceding propositions; but Euclid has preferred the method of exhibiting, in the demonstrations of the second book, the equality of the spaces compared.

In the corollary to Prop. xlv. Book I, it is stated that a parallelogram which has one right angle, has all its angles right angles. By applying this corollary, the demonstration of Prop. iv. may be considerably shortened.

If the two parts of the line be equal, then the square on the whole line is equal to four times the square on half the line.

Also, if a line be divided into any three parts, the square on the whole line is equal to the squares on the three parts, and twice the rectangles contained by every two parts.

Prop. iv. Algebraically. (fig. Prop. iv.)

Let the line  $AB$  contain  $a$  linear units, and the parts of it  $AC$  and  $BC$ ,  $m$  and  $n$  linear units respectively.

$$\begin{aligned}\text{Then } a &= m + n, \\ \text{squaring these equals, } \therefore a^2 &= (m + n)^2, \\ \text{or } a^2 &= m^2 + 2mn + n^2.\end{aligned}$$

That is, if a number be divided into any two parts, the square of the number is equal to the squares of the two parts together with twice the product of the two parts.

From Euc. II. 4, may be deduced a proof of Euc. I. 47. In the fig. take  $DL$  on  $DE$ , and  $EM$  on  $EB$ ; each equal to  $BC$ , and join  $CH$ ,  $HL$ ,  $LM$ ,  $MC$ . Then the figure  $HLMC$  is a square, and the four triangles  $CAH$ ,  $HDL$ ,  $LEM$ ,  $MBC$  are equal to one another, and together are equal to the two rectangles  $AG$ ,  $GE$ .

Now  $AG$ ,  $GE$ ,  $FH$ ,  $CK$  are together equal to the whole figure  $ADEB$ ; and  $HLMC$ , with the four triangles  $CAH$ ,  $HDL$ ,  $LEM$ ,  $MBC$  also make up the whole figure  $ADEB$ ;

Hence  $AG$ ,  $GE$ ,  $FH$ ,  $CK$  are equal to  $HLMC$  together with the four triangles:

but  $AG$ ,  $GE$  are equal to the four triangles.

wherefore  $FH$ ,  $CK$  are equal to  $HLMC$ ,

that is, the squares on  $AC$ ,  $AB$  are together equal to the square on  $CH$ .

Prop. v. It must be kept in mind, that the sum of two straight lines in Geometry, means the straight line formed by joining the two lines together, so that both may be in the same straight line.

The following simple properties respecting the equal and unequal division of a line are worthy of being remembered.

I. Since  $AB = 2BC = 2(BD + DC) = 2BD + 2DC$ . (fig. Prop. v.)

$$\text{and } AB = AD + DB;$$

$$\therefore 2CD + 2DB = AD + DB,$$

and by subtracting  $2DB$  from these equals,

$$\therefore 2CD = AD - DB,$$

$$\text{and } CD = \frac{1}{2}(AD - DB).$$

That is, if a line  $AB$  be divided into two equal parts in  $C$ , and into two unequal parts in  $D$ , the part  $CD$  of the line between the points of section is equal to half the difference of the unequal parts  $AD$  and  $DB$ .

II. Here  $AD = AC + CD$ , the sum of the unequal parts, (fig. Prop. v.)

and  $DB = AC - CD$  their difference.

Hence by adding these equals together,

$$\therefore AD + DB = 2AC,$$

or the sum and difference of two lines  $AC$ ,  $CD$ , are together equal to twice the greater line.

And the halves of these equals are equal,

$$\therefore \frac{1}{2}.AD + \frac{1}{2}.DB = AC,$$

or, half the sum of two unequal lines  $AC$ ,  $CD$  added to half their difference is equal to the greater line  $AC$ .

III. Again, since  $AD = AC + CD$ , and  $DB = AC - CD$ , by subtracting these equals,

$$\therefore AD - DB = 2CD,$$

or, the difference between the sum and difference of two unequal lines is equal to twice the less line.

And the halves of these equals are equal,

$$\therefore \frac{1}{2}.AD - \frac{1}{2}.DB = CD,$$

or, half the difference of two lines subtracted from half their sum is equal to the less of the two lines.

IV. Since  $AC - CD = DB$  the difference,

$$\therefore AC = CD + DB,$$

and adding  $CD$  the less to each of these equals,

$$\therefore AC + CD = 2CD + DB,$$

or, the sum of two unequal lines is equal to twice the less line together with the difference between the lines.

Prop. v. Algebraically.

Let  $AB$  contain  $2a$  linear units,

its half  $BC$  will contain  $a$  linear units.

And let  $CD$  the line between the points of section contain  $m$  linear units. Then  $AD$  the greater of the two unequal parts, contains  $a + m$  linear units; and  $DB$  the less contains  $a - m$  units.

Also  $m$  is half the difference of  $a + m$  and  $a - m$ ;

$$\therefore (a + m)(a - m) = a^2 - m^2,$$

to each of these equals add  $m^2$ ;

$$\therefore (a + m)(a - m) + m^2 = a^2.$$

That is, if a number be divided into two equal parts, and also into two unequal parts, the product of the unequal parts together with the square of half their difference, is equal to the square of half the number.

Bearing in mind that  $AC$ ,  $CD$  are respectively half the sum and half the difference of the two lines  $AD$ ,  $DB$ ; the corollary to this proposition may be expressed in the following form: "The rectangle contained by two straight lines is equal to the difference on the squares of half their sum and half their difference."

The rectangle contained by  $AD$  and  $DB$ , and the square on  $BC$  are each bounded by the same extent of line, but the spaces enclosed differ by the square on  $CD$ .

A given straightline is said to be *produced* when it has its length increased in either direction, and the increase it receives, is called the *part produced*.

If a point be taken in a line or in a line produced, the line is said to be divided *internally* or *externally*, and the distances of the point from



the ends of the line are called the internal or external segments of the line, according as the point of section is in the line or the line produced.

Prop. VI. Algebraically.

Let  $AB$  contain  $2a$  linear units, then its half  $BC$  contains  $a$  units; and let  $BD$  contain  $m$  units.

Then  $AD$  contains  $2a + m$  units,

and  $\therefore (2a + m)m = 2am + m^2$ ;

to each of these equals add  $a^2$ ,

$$\therefore (2a + m)m + a^2 = a^2 + 2am + m^2.$$

But  $a^2 + 2am + m^2 = (a + m)^2$ ,

$$\therefore (2a + m)m + a^2 = (a + m)^2.$$

That is, If a number be divided into two equal numbers, and another number be added to the whole and to one of the parts; the product of the whole number thus increased and the other number, together with the square of half the given number, is equal to the square of the number which is made up of half the given number increased.

The algebraical results of Prop. V. and Prop. VI. are identical, as it is obvious that the difference of  $a + m$  and  $a - m$  in Prop. V. is equal to the difference of  $2a + m$  and  $m$  in Prop. VI. and one algebraical result expresses the truth of both propositions.

This arises from the two ways in which the difference between two unequal lines may be represented geometrically, when they are in the same direction.

In the diagram (fig. to Prop. V.), the difference  $DB$  of the two unequal lines  $AC$  and  $CD$  is exhibited by producing the less line  $CD$ , and making  $CB$  equal to  $AC$  the greater.

Then the part produced  $DB$  is the difference between  $AC$  and  $CD$ ,

for  $AC$  is equal to  $CB$ , and taking  $CD$  from each,

the difference of  $AC$  and  $CD$  is equal to the difference of  $CB$  and  $CD$ .

In the diagram (fig. to Prop. VI.), the difference  $DB$  of the two unequal lines  $CD$  and  $CA$  is exhibited by cutting off from  $CD$  the greater, a part  $CB$  equal to  $CA$  the less.

Prop. VII. Either of the two parts  $AC$ ,  $CB$  of the line  $AB$  may be taken: and it is equally true, that the squares on  $AB$  and  $AC$  are equal to twice the rectangle  $AB$ ,  $AC$ , together with the square on  $BC$ .

Prop. VII. Algebraically.

Let  $AB$  contain  $a$  linear units, and let the parts  $AC$  and  $CB$  contain  $m$  and  $n$  linear units respectively.

Then  $a = m + n$ ;

squaring these equals,

$$\therefore a^2 = m^2 + 2mn + n^2,$$

add  $n^2$  to each of these equals,

$$\therefore a^2 + n^2 = m^2 + 2mn + 2n^2.$$

$$\text{But } 2mn + 2n^2 = 2(m + n)n = 2an,$$

$$\therefore a^2 + n^2 = m^2 + 2an.$$

That is, If a number be divided into any two parts, the squares of the whole number and of one of the parts are equal to twice the product of the whole number and that part, together with the square of the other part.

Prop. VIII. As in Prop. VII. either part of the line may be taken, and it is also true in this Proposition, that four times the rectangle con-

tained by  $AB, AC$  together with the square on  $BC$ , is equal to the square on the straight line made up of  $AB$  and  $AC$  together.

The truth of this proposition may be deduced from *Euc. II. 4* and *7*.

For the square on  $AD$  (*fig. Prop. 8.*) is equal to the squares on  $AB, BD$ , and twice the rectangle  $AB, BD$ ; (*Euc. II. 4.*) or the squares on  $AB, BC$ , and twice the rectangle  $AB, BC$ , because  $BC$  is equal to  $BD$ : and the squares on  $AB, BC$  are equal to twice the rectangle  $AB, BC$  with the square on  $AC$ : (*Euc. II. 7.*) therefore the square on  $AD$  is equal to four times the rectangle  $AB, BC$  together with the square on  $AC$ .

*Prop. VIII. Algebraically.*

Let the whole line  $AB$  contain  $a$  linear units of which the parts  $AC, CB$  contain  $m, n$  units respectively.

Then  $m + n = a$ ,

and subtracting or taking  $n$  from each,

$$\therefore m = a - n,$$

squaring these equals,

$$\therefore m^2 = a^2 - 2an + n^2,$$

and adding  $4an$  to each of these equals,

$$\therefore 4an + m^2 = a^2 + 2an + n^2.$$

$$\text{But } a^2 + 2an + n^2 = (a + n)^2,$$

$$\therefore 4an + m^2 = (a + n)^2.$$

That is, If a number be divided into any two parts, four times the product of the whole number and one of the parts, together with the square of the other part, is equal to the square of the number made of the whole and the part first taken.

*Prop. VIII.* may be put under the following form: The square on the sum of two lines exceeds the square on their difference, by four times the rectangle contained by the lines.

*Prop. IX.* The demonstration of this proposition may be deduced from *Euc. II. 4* and *7*.

For (*Euc. II. 4.*) the square on  $AD$  is equal to the squares on  $AC, CD$  and twice the rectangle  $AC, CD$ ; (*fig. Prop. 9.*) and adding the square on  $DB$  to each, therefore the squares on  $AD, DB$  are equal to the squares on  $AC, CD$  and twice the rectangle  $AC, CD$  together with the square on  $DB$ ; or to the squares on  $BC, CD$  and twice the rectangle  $BC, CD$  with the square on  $DB$ , because  $BC$  is equal to  $AC$ .

But the squares on  $BC, CD$  are equal to twice the rectangle  $BC, CD$ , with the square on  $DB$ . (*Euc. II. 7.*)

Wherefore the squares on  $AD, DB$  are equal to twice the squares on  $BC$  and  $CD$ .

*Prop. IX. Algebraically.*

Let  $AB$  contain  $2a$  linear units, its half  $AC$  or  $BC$  will contain  $a$  units; and let  $CD$  the line between the points of section contain  $m$  units.

Also  $AD$  the greater of the two unequal parts contains  $a + m$  units,

and  $DB$  the less contains  $a - m$  units.

$$\text{Then } (a + m)^2 = a^2 + 2am + m^2,$$

$$\text{and } (a - m)^2 = a^2 - 2am + m^2.$$

Hence by adding these equals,

$$\therefore (a + m)^2 + (a - m)^2 = 2a^2 + 2m^2.$$

That is, If a number be divided into two equal parts, and also into two unequal parts, the sum of the squares of the two unequal parts is equal to twice the square of half the number itself, and twice the square of half the difference of the unequal parts.

The proof of Prop. x. may be deduced from Euc. II. 4, 7, as Prop. ix. Prop. x. Algebraically.

Let the line  $AB$  contain  $2a$  linear units, of which its half  $AC$  or  $CB$  will contain  $a$  units;

and let  $BD$  contain  $m$  units.

Then the whole line and the part produced will contain  $2a + m$  units, and half the line and the part produced will contain  $a + m$  units,

$$\therefore (2a + m)^2 = 4a^2 + 4am + m^2,$$

add  $m^2$  to each of these equals,

$$\therefore (2a + m)^2 + m^2 = 4a^2 + 4am + 2m^2.$$

Again,  $(a + m)^2 = a^2 + 2am + m^2$ ,

add  $a^2$  to each of these equals,

$$\therefore (a + m)^2 + a^2 = 2a^2 + 2am + m^2,$$

and doubling these equals,

$$\therefore 2(a + m)^2 + 2a^2 = 4a^2 + 4am + 2m^2.$$

$$\text{But } (2a + m)^2 + m^2 = 4a^2 + 4am + 2m^2.$$

$$\text{Hence } \therefore (2a + m)^2 + m^2 = 2a^2 + 2(a + m)^2.$$

That is, If a number be divided into two equal parts, and the whole number and one of the parts be increased by the addition of another number, the squares of the whole number thus increased, and of the number by which it is increased, are equal to double the squares of half the number, and of half the number increased.

The algebraical results of Prop. ix, and Prop. x, are identical, (the enunciations of the two Props. arising, as in Prop. v, and Prop. vi, from the two ways of exhibiting the difference between two lines); and both may be included under the following proposition: The square on the sum of two lines and the square on their difference, are together equal to double the sum of the squares on the two lines.

Prop. xi. Two series of lines, one series decreasing and the other series increasing in magnitude, and each line divided in the same manner may be found by means of this proposition.

(1) To find the decreasing series.

In the fig. Euc. II. 11,  $AB = AH + BH$ ,

and since  $AB \cdot BH = AH^2$ ,  $\therefore (AH + BH) \cdot BH = AH^2$ ,

$$\therefore BH^2 = AH^2 - AH \cdot BH = AH \cdot (AH - BH).$$

If now in  $HA$ ,  $HL$  be taken equal to  $BH$ ,

$$\text{then } HL^2 = AH(AH - HL), \text{ or } AH \cdot AL = HL^2:$$

that is,  $AH$  is divided in  $L$ , so that the rectangle contained by the whole line  $AH$  and one part, is equal to the square on the other part  $HL$ . By a similar process,  $HL$  may be so divided; and so on, by always taking from the greater part of the divided line, a part equal to the less.

(2) To find the increasing series.

From the fig. it is obvious that  $CF \cdot FA = CA^2$ ,

Hence  $CF$  is divided in  $A$ , in the same manner as  $AB$  is divided in  $H$ , by adding  $AF$  a line equal to the greater segment, to the given line  $CA$

or  $AB$ . And by successively adding to the last line thus divided, its greater segment, a series of lines increasing in magnitude may be found similarly divided to  $AB$ .

It may also be shewn that the squares on the whole line and on the less segment are equal to three times the square on the greater segment. (Euc. XIII. 4.)

To solve Prop. XI, algebraically, or to find the point  $H$  in  $AB$  such that the rectangle contained by the whole line  $AB$  and the part  $HB$  shall be equal to the square on the other part  $AH$ .

Let  $AB$  contain  $a$  linear units, and  $AH$  one of the unknown parts contain  $x$  units,

then the other part  $HB$  contains  $a - x$  units.

And  $\therefore a(a - x) = x^2$ , by the problem,

or  $x^2 + ax = a^2$ , a quadratic equation.

$$\text{Whence } x = \frac{\pm a \sqrt{5} - a}{2}.$$

The former of these values of  $x$  determines the point  $H$ .

So that  $x = \frac{\sqrt{5} - 1}{2} \cdot AB = AH$ , one part,

and  $a - x = a - AH = \frac{3 - \sqrt{5}}{2} \cdot AB = HB$ , the other part.

It may be observed, that the parts  $AH$  and  $HB$  cannot be numerically expressed by any rational number. Approximation to their true values in terms of  $AB$ , may be made to any required degree of accuracy, by extending the extraction of the square root of 5 to any number of decimals.

To ascertain the meaning of the other result  $x = -\frac{\sqrt{5} + 1}{2} \cdot a$ .

In the equation  $a(a - x) = x^2$ ,

for  $x$  write  $-x$ , then  $a(a + x) = x^2$ ,

which when translated into words gives the following problem.

To find the length to which a given line must be produced so that the rectangle contained by the given line and the line made up of the given line and the part produced, may be equal to the square on the part produced.

Or, the problem may also be expressed as follows:

To find two lines having a given difference, such that the rectangle contained by the difference and one of them may be equal to the square on the other.

It may here be remarked, that Prop. XI. Book II, affords a simple Geometrical construction for a quadratic equation.

Prop. XII. Algebraically.

Assuming the truth of Euc. I. 47.

Let  $BC$ ,  $CA$ ,  $AB$  contain  $a$ ,  $b$ ,  $c$  linear units respectively,

and let  $CD$ ,  $DA$ , contain  $m$ ,  $n$  units,

then  $BD$  contains  $a + m$  units.

And therefore,  $c^2 = (a + m)^2 + n^2$ , from the right-angled triangle  $ABD$ ,

also  $b^2 = m^2 + n^2$  from  $ACD$ ;

$$\therefore c^2 - b^2 = (a + m)^2 - m^2$$

$$= a^2 + 2am + m^2 - m^2$$

$$= a^2 + 2am,$$

$$\therefore c^2 = b^2 + a^2 + 2am,$$

that is,  $c^2$  is greater than  $b^2 + a^2$  by  $2am$ .

Prop. XIII. Case II. may be proved more simply as follows.

Since  $BD$  is divided into two parts in the point  $D$ , therefore the squares on  $CB$ ,  $BD$  are equal to twice the rectangle contained by  $CB$ ,  $BD$  and the square on  $CD$ ; (II. 7.)

add the square on  $AD$  to each of these equals;

therefore the squares on  $CB$ ,  $BD$ ,  $DA$  are equal to twice the rectangle  $CB$ ,  $BD$ , and the squares on  $CD$  and  $DA$ ,

but the squares on  $BD$ ,  $DA$  are equal to the square on  $AB$ , (I. 47.)

and the squares on  $CD$ ,  $DA$  are equal to the square on  $AC$ ,

therefore the squares on  $CB$ ,  $BA$  are equal to the square on  $AC$ , and twice the rectangle  $CB$ ,  $BD$ . That is, &c.

Prop. XIII. Algebraically.

Let  $BC$ ,  $CA$ ,  $AB$  contain respectively  $a$ ,  $b$ ,  $c$  linear units, and let  $BD$  and  $AD$  also contain  $m$  and  $n$  units.

Case I. Then  $DC$  contains  $a - m$  units.

Therefore  $c^2 = n^2 + m^2$  from the right-angled triangle  $ABD$ ,

and  $b^2 = n^2 + (a - m)^2$  from  $ADC$ ;

$$\therefore c^2 - b^2 = m^2 - (a - m)^2$$

$$= m^2 - a^2 + 2am - m^2$$

$$= -a^2 + 2am,$$

$$\therefore a^2 + c^2 = b^2 + 2am,$$

$$\text{or } b^2 + 2am = a^2 + c^2,$$

that is,  $b^2$  is less than  $a^2 + c^2$  by  $2am$ .

Case II.  $DC = m - a$  units,

$\therefore c^2 = m^2 + n^2$  from the right-angled triangle  $ABD$ ,

and  $b^2 = (m - a)^2 + n^2$  from  $ACD$ ,

$$\therefore c^2 - b^2 = m^2 - (m - a)^2,$$

$$= m^2 - m^2 + 2am - a^2$$

$$= 2am - a^2,$$

$$\therefore a^2 + c^2 = b^2 + 2am,$$

$$\text{or } b^2 + 2am = a^2 + c^2,$$

that is,  $b^2$  is less than  $a^2 + c^2$  by  $2am$ .

Case III. Here  $m$  is equal to  $a$ .

And  $b^2 + a^2 = c^2$ , from the right-angled triangle  $ABC$ .

Add to each of these equals  $a^2$ ,

$$\therefore b^2 + 2a^2 = c^2 + a^2,$$

that is,  $b^2$  is less than  $c^2 + a^2$  by  $2a^2$ , or  $2aa$ .

These two propositions, Euc. II. 12, 13, with Euc. I. 47, exhibit the relations which subsist between the sides of an obtuse-angled, an acute-angled, and right-angled triangle respectively.

## NOTE ON THE ABBREVIATIONS AND ALGEBRAICAL SYMBOLS EMPLOYED IN GEOMETRY.

THE ancient Geometry of the Greeks admitted no symbols besides the diagrams and ordinary language. In later times, after symbols of operation had been devised by writers on Algebra, they were very soon adopted and employed on account of their brevity and convenience, in writings purely geometrical. Dr. Barrow was one of the first who introduced algebraical symbols into the language of Elementary Geometry, and distinctly states in the preface to his Euclid, that his object is "to content the desires of those who are delighted more with symbolical than verbal demonstrations." As algebraical symbols are employed in almost all works on the mathematics, whether geometrical or not, it seems proper in this place to give some brief account of the marks which may be regarded as the alphabet of symbolical language.

The mark = was first used by Robert Recorde, in his treatise on Algebra entitled, "The Whetstone of Witte," 1557. He remarks; "And to avoide the tedious repetition of these woordes : *is equalle to* : I will sette as I doe often in woorke use, a paire of paraleles, or Gemowe lines of one lengthe, thus : =, bicause noe 2 thynges can be more equalle." It was employed by him as simply affirming the equality of two numerical or algebraical expressions. Geometrical equality is not exactly the same as numerical equality, and when this symbol is used in geometrical reasonings, it must be understood as having reference to pure geometrical equality.

The signs of relative magnitude, > meaning, *is greater than*, and <, *is less than*, were first introduced into algebra by Thomas Harriot, in his "Artis Analyticæ Praxis," which was published after his death in 1631.

The signs + and - were first employed by Michael Stifel, in his "Arithmetica Integra," which was published in 1544. The sign + was employed by him for the word *plus*, and the sign -, for the word *minus*. These signs were used by Stifel strictly as the arithmetical or algebraical signs of addition and subtraction.

The sign of multiplication  $\times$  was first introduced by Oughtred in his "Clavis Mathematica," which was published in 1631. In algebraical multiplication he either connects the letters which form the factors of a product by the sign  $\times$ , or writes them as words without any sign or mark between them, as had been done before by Harriot, who first introduced the small letters to designate known and unknown quantities. However concise and convenient the notation  $AB \times BC$  or  $AB \cdot BC$  may be in practice for "*the rectangle contained by the lines AB and BC*"; the student is cautioned against the use of it, in the early part of his geometrical studies, as its use is likely to occasion a misapprehension of Euclid's meaning, by confounding the idea of Geometrical equality with that of Arithmetical equality. Later writers on Geometry who employed the Latin language, explained the notation  $AB \times BC$ , by "*AB ductum in BC*"; that is, if the line  $AB$  be carried along the line  $BC$  in a normal position to it, until it come to the end  $C$ , it will then form with  $BC$ , the rectangle contained by  $AB$  and  $BC$ . Dr. Barrow sometimes expresses "*the rectangle contained by AB and BC*" by "*the rectangle ABC*."

Michael Stifel was the first who introduced integral exponents to denote the powers of algebraical symbols of quantity, for which he employed capital letters. Vieta afterwards used the vowels to denote known, and the consonants, unknown quantities, but used words to designate the

powers. Simon Stevin, in his treatise on Algebra, which was published in 1605, improved the notation of Stifel, by placing the figures that indicated the powers within small circles. Peter Ramus adopted the initial letters  $l, q, c, b$  of *latus, quadratus, cubus, biquadratus*, as the notation of the first four powers. Harriot exhibited the different powers of algebraical symbols by repeating the symbol, two, three, four, &c. times, according to the order of the power. Descartes restored the numerical exponents of powers, placing them at the right of the numbers, or symbols of quantity, as at the present time. Dr. Barrow employed the notation  $ABq$ , for "*the square on the line  $AB$* ," in his edition of Euclid. The notations  $AB^2, AB^3$ , for "*the square and cube on the line whose extremities are  $A$  and  $B$* ," as well as  $AB \times BC$ , for "*the rectangle contained by  $AB$  and  $BC$* ," are used as abbreviations in almost all works on the Mathematics, though not wholly consistent with the algebraical notations  $a^2$  and  $a^3$ .

The symbol  $\sqrt{\phantom{x}}$ , being originally the initial letter of the word *radix*, was first used by Stifel to denote the square root of the number, or of the symbol, before which it is placed.

The Hindus, in their treatises on Algebra, indicated the ratio of two numbers, or of two algebraical symbols, by placing one above the other, without any line of separation. The line was first introduced by the Arabians, from whom it passed to the Italians, and from them to the rest of Europe. This notation has been employed for the expression of geometrical ratios by almost all writers on the Mathematics, on account of its great convenience. Oughtred first used points to indicate proportion; thus,  $a : b :: c : d$ , means that  $a$  bears the same proportion to  $b$ , as  $c$  does to  $d$ .

## QUESTIONS ON BOOK II.

1. Is *rectangle* the same as *rectus angulus*? Explain the distinction, and give the corresponding Greek terms.

2. What is meant by *the sum* of two, or more than two straight lines in Geometry?

3. Is there any difference between the straight lines by which a rectangle is said to be contained, and those by which it is bounded?

4. Define a *gnomon*. How many *gnomons* appear from the same construction in the same rectangle? Find the difference between them.

5. What axiom is assumed in proving the first eight propositions of the Second Book of Euclid?

6. Of equal squares and equal rectangles, which must necessarily coincide?

7. How may a rectangle be dissected so as to form an equivalent rectangle of any proposed length?

8. When the adjacent sides of a rectangle are commensurable, the area of the rectangle is properly represented by the product of the number of units in two adjacent sides of the rectangle. Illustrate this by considering the case when the two adjacent sides contain 3 and 4 units respectively, and distinguish between the units of the factors and the units of the product. Shew generally that a rectangle whose adjacent sides are represented by the integers  $a$  and  $b$ , is represented by  $ab$ . Also shew, that in the same sense,

the rectangle is represented by  $\frac{ab}{mn}$ , if the sides be represented by  $\frac{a}{m}, \frac{b}{n}$ .

9. Why may not Algebraical or Arithmetical proofs be substituted (as being shorter) for the demonstrations of the Propositions in the Second Book of Euclid?

10. In what sense is the *area* of a triangle said to be equal to half the product of its base and its altitude? What two propositions of Euclid may be adduced to prove it?

11. How do you shew that the area of a rhombus is equal to half the rectangle contained by the diagonals?

12. How may a rule be deduced for finding a numerical expression for the area of any parallelogram, when two adjacent sides are given?

13. The area of a trapezium which has two of its sides parallel is equal to that of a rectangle contained by its altitude and half the sum of its parallel sides. What propositions of the First and Second Books of Euclid are employed to prove this? Of what service is the above in the measurement of fields with irregular borders?

14. From what propositions of Euclid may be deduced the following rule for finding the area of any quadrilateral figure:—"Multiply the sum of the perpendiculars drawn from opposite angles of the figure upon the diagonal joining the other two angles, and take half the product."

15. In Euclid, II. 3, where must be the point of division of the line, so that the rectangle contained by the two parts may be a maximum? Exemplify in the case where the line is 12 inches long.

16. How may the demonstration of Euclid II. 4, be legitimately shortened? Give the Algebraical proof, and state on what suppositions it can be regarded as a proof.

17. Shew that the proof of Euc. II. 4, can be deduced from the two previous propositions without any geometrical construction.

18. Shew that if the two complements be together equal to the two squares, the given line is bisected.

19. If the line  $AB$ , as in Euc. II. 4, be divided into any three parts, enunciate and prove the analogous proposition.

20. Prove geometrically that if a straight line be trisected, the square on the whole line equals nine times the square on a third part of it.

21. Deduce from Euc. II. 4, a proof of Euc. I. 47.

22. If a straight line be divided into two parts, when is the rectangle contained by the parts, *the greatest possible*? and when is the sum of the squares of the parts, *the least possible*?

23. Shew that if a line be divided into two equal parts and into two unequal parts; the part of the line between the points of section is equal to half the difference of the unequal parts.

24. If half the sum of two unequal lines be increased by half their difference, the sum will be equal to the greater line: and if the sum of two lines be diminished by half their difference, the remainder will be equal to the less line.

25. Explain what is meant by the *internal* and *external segments* of a line; and show that the sum of the external segments of a line or the difference of the internal segments is double the distance between the points of section and bisection of the line.

26. Shew how Euc. II. 6, may be deduced immediately from the preceding Proposition.

27. Prove Geometrically that the squares on the sum and difference of two lines are equal to twice the squares on the lines themselves.

28. A given rectangle is divided by two straight lines into four rectangles. Given the areas of the two which have not common sides: find the areas of the other two.



29. In how many ways may the difference of two lines be exhibited? Enunciate the propositions in Book II, which depend on that circumstance.

30. How may a series of lines be found similarly divided to the line  $AB$  in Euc. II. 11?

31. Divide Algebraically a given line ( $a$ ) into two parts, such that the rectangle contained by the whole and one part may be equal to the square of the other part. Deduce Euclid's construction from one solution, and explain the other.

32. Given the lesser segment of a line, divided as in Euc. II. 11, find the greater.

33. Enunciate the Arithmetical theorems expressed by the following Algebraical formulæ,

$$(a+b)^2 = a^2 + 2ab + b^2; a^2 - b^2 = (a+b)(a-b); (a-b)^2 = a^2 - 2ab + b^2,$$

and state the corresponding Geometrical propositions.

34. Shew that the first of the Algebraical propositions,

$$(a+x)(a-x) + x^2 = a^2; (a+x)^2 + (a-x)^2 = 2a^2 + 2x^2,$$

is equivalent to the two propositions V. and VI., and the second of them, to the two propositions IX. and X. of the Second Book of Euclid.

35. Prove Euc. II. 12, when the perpendicular  $BE$  is drawn from  $B$  on  $AC$  produced to  $E$ , and shew that the rectangle  $BC, CD$  is equal to the rectangle  $AC, CE$ .

36. Include the first two cases of Euc. II. 13, in one proof.

37. In the second case of Euc. II. 13, draw a perpendicular  $CE$  from the obtuse angle  $C$  upon the side  $AB$ , and prove that the square on  $AB$  is equal to the rectangle  $AB, AE$  together with the rectangle  $BC, BD$ .

38. Enunciate Euc. II. 13, and give an Algebraical or Arithmetical proof of it.

39. The sides of a triangle are as 3, 4, 5. Determine whether the angles between 3, 4; 4, 5; and 3, 5; respectively are greater than, equal to, or less than, a right angle.

40. Two sides of a triangle are 4 and 5 inches in length, if the third side be  $6\frac{1}{2}$  inches, the triangle is acute-angled, but if it be  $6\frac{1}{3}$  inches, the triangle is obtuse-angled.

41. A triangle has its sides 7, 8, 9 units respectively; a strip of breadth 2 units being taken off all round from the triangle, find the area of the remainder.

42. If the original figure, Euc. II. 14, were a right-angled triangle, whose sides were represented by 8 and 9, what number would represent the side of a square of the same area? Shew that the perimeter of the square is less than the perimeter of the triangle.

43. If the sides of a rectangle are 8 feet and 2 feet, what is the side of the equivalent square?

44. "All plane rectilineal figures admit of quadrature." Point out the succession of steps by which Euclid establishes the truth of this proposition.

45. Explain the construction (without proof) for making a square equal to a plane polygon.

46. Shew from Euc. II. 14, that any algebraical surd as  $\sqrt{a}$  can be represented by a line, if the unit be a line.

47. Could any of the propositions of the Second Book be made *collaries* to other propositions, with advantage? Point out any such propositions, and give your reasons for the alterations you would make.

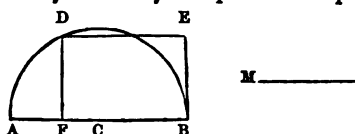
## GEOMETRICAL EXERCISES ON BOOK II.

### PROPOSITION I. PROBLEM.

*Divide a given straight line into two parts such, that their rectangle may be equal to a given square; and determine the greatest square which the rectangle can equal.*

Let  $AB$  be the given straight line, and let  $M$  be the side of the given square.

It is required to divide the line  $AB$  into two parts, so that the rectangle contained by them may be equal to the square on  $M$ .



Bisect  $AB$  in  $C$ , with center  $C$ , and radius  $CA$  or  $CB$ , describe the semicircle  $ADB$ .

At the point  $B$  draw  $BE$  at right angles to  $AB$  and equal to  $M$ .

Through  $E$ , draw  $ED$  parallel to  $AB$  and cutting the semicircle in  $D$ ;

and draw  $DF$  parallel to  $EB$  meeting  $AB$  in  $F$ .

Then  $AB$  is divided in  $F$ , so that the rectangle  $AF, FB$  is equal to the square on  $M$ . (II. 14.)

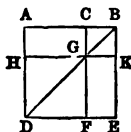
The square will be the greatest, when  $ED$  touches the semicircle, or when  $M$  is equal to half of the given line  $AB$ .

### PROPOSITION II. THEOREM.

*The square on the excess of one straight line above another is less than the squares on the two lines by twice their rectangle.*

Let  $AB, BC$  be the two straight lines, whose difference is  $AC$ .

Then the square on  $AC$  is less than the squares on  $AB$  and  $BC$  by twice the rectangle contained by  $AB$  and  $BC$ .



Constructing as in Prop. 4. Book II.

Because the complement  $AG$  is equal to  $GE$ ,

add to each  $CK$ ,

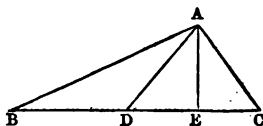
therefore the whole  $AK$  is equal to the whole  $CE$ ;

and  $AK, CE$  together are double of  $AK$ ;  
 but  $AK, CE$  are the gnomon  $AKF$  and  $CK$ ,  
 and  $AK$  is the rectangle contained by  $AB, BC$ ;  
 therefore the gnomon  $AKF$  and  $CK$ ,  
 are equal to twice the rectangle  $AB, BC$ ,  
 but  $AE, CK$  are equal to the squares on  $AB, BC$ ;  
 taking the former equals from these equals,  
 therefore the difference of  $AE$  and the gnomon  $AKF$  is equal to  
 the difference between the squares on  $AB, BC$ , and twice the rectangle  
 $AB, BC$ ;  
 but the difference  $AE$  and the gnomon  $AKF$  is the figure  $HF$ ,  
 which is equal to the square on  $AC$ .  
 Wherefore the square on  $AC$  is equal to the difference between the  
 squares on  $AB, BC$ , and twice the rectangle  $AB, BC$ .

### PROPOSITION III. THEOREM.

*In any triangle the squares on the two sides are together double of the squares on half the base and on the straight line joining its bisection with the opposite angle.*

Let  $ABC$  be a triangle, and  $AD$  the line drawn from the vertex  $A$  to the bisection  $D$  of the base  $BC$ .



From  $A$  draw  $AE$  perpendicular to  $BC$ .

Then, in the obtuse-angled triangle  $ABD$ , (II. 12.);  
 the square on  $AB$  exceeds the squares on  $AD, DB$ , by twice the  
 rectangle  $BD, DE$ :

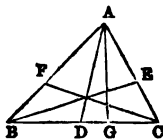
and in the acute-angled triangle  $ADC$ , (II. 13.);  
 the square on  $AC$  is less than the squares on  $AD, DC$ , by twice  
 the rectangle  $CD, DE$ :

wherefore, since the rectangle  $BD, DE$  is equal to the rectangle  $CD, DE$ ; it follows that the squares on  $AB, AC$  are double of the squares on  $AD, DB$ .

### PROPOSITION IV. THEOREM.

*If straight lines be drawn from each angle of a triangle bisecting the opposite sides, four times the sum of the squares on these lines is equal to three times the sum of the squares on the sides of the triangle.*

Let  $ABC$  be any triangle, and let  $AD, BE, CF$  be drawn from  $A, B, C$ , to  $D, E, F$ , the bisections of the opposite sides of the triangle: draw  $AG$  perpendicular to  $BC$ .



Then the square on  $AB$  is equal to the squares on  $BD$ ,  $DA$  together with twice the rectangle  $BD$ ,  $DG$ , (II. 12.)

and the square on  $AC$  is equal to the squares on  $CD$ ,  $DA$  diminished by twice the rectangle  $CD$ ,  $DG$ ; (II. 13.)

therefore the squares on  $AB$ ,  $AC$  are equal to twice the square on  $BD$ , and twice the square on  $AD$ ; for  $DC$  is equal to  $BD$ :

and twice the squares on  $AB$ ,  $AC$  are equal to the square on  $BC$ , and four times the square on  $AD$ : for  $BC$  is twice  $BD$ .

Similarly, twice the squares on  $AB$ ,  $BC$  are equal to the square on  $AC$ , and four times the square on  $BE$ :

also twice the squares on  $BC$ ,  $CA$  are equal to the square on  $AB$ , and four times the square on  $FC$ :

hence, by adding these equals,

four times the squares on  $AB$ ,  $AC$ ,  $BC$  are equal to four times the squares on  $AD$ ,  $BE$ ,  $CF$  together with the squares on  $AB$ ,  $AC$ ,  $BC$ :

and taking the squares on  $AB$ ,  $AC$ ,  $BC$  from these equals,

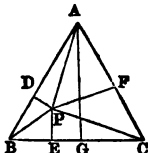
therefore three times the squares on  $AB$ ,  $AC$ ,  $BC$  are equal to four times the squares on  $AD$ ,  $BE$ ,  $CF$ .

#### PROPOSITION V. THEOREM.

*The sum of the perpendiculars let fall from any point within an equilateral triangle, will be equal to the perpendicular let fall from one of its angles upon the opposite side. Is this proposition true when the point is in one of the sides of the triangle? In what manner must the proposition be enunciated when the point is without the triangle?*

Let  $ABC$  be an equilateral triangle, and  $P$  any point within it: and from  $P$  let fall  $PD$ ,  $PE$ ,  $PF$  perpendiculars on the sides  $AB$ ,  $BC$ ,  $CA$  respectively, also from  $A$  let fall  $AG$  perpendicular on the base  $BC$ .

Then  $AG$  is equal to the sum of  $PD$ ,  $PE$ ,  $PF$ .



From  $P$  draw  $PA$ ,  $PB$ ,  $PC$  to the angles  $A$ ,  $B$ ,  $C$ .

Then the triangle  $ABC$  is equal to the three triangles  $PAB$ ,  $PBC$ ,  $PCA$ .

But since every rectangle is double of a triangle of the same base and altitude, (I. 41.)

therefore the rectangle  $AG, BC$ , is equal to the three rectangles  $AB, PD$ ;  $AC, PF$  and  $BC, PE$ .

Whence the line  $AG$  is equal to the sum of the lines  $PD, PE, PF$ .  
If the point  $P$  fall on one side of the triangle, or coincide with  $E$ :

then the triangle  $ABC$  is equal to the two triangles  $APC, BPA$ :  
whence  $AG$  is equal to the sum of the two perpendiculars  $PD, PF$ .

If the point  $P$  fall without the base  $BC$  of the triangle:

then the triangle  $ABC$  is equal to the difference between the sum of the two triangles  $APC, BPA$ , and the triangle  $PCB$ .

Whence  $AG$  is equal to the difference between the sum of  $PD, PF$ , and  $PE$ .

### I.

6. If the straight line  $AB$  be divided into two unequal parts in  $D$ , and into two unequal parts in  $E$ , the rectangle contained by  $AE, EB$ , will be greater or less than the rectangle contained by  $AD, DB$ , according as  $E$  is nearer to, or further from, the middle point of  $AB$ , than  $D$ .

7. Produce a given straight line in such a manner that the square on the whole line thus produced, shall be equal to twice the square on the given line.

8. If  $AB$  be the line so divided in the points  $C$  and  $D$ , (fig. Euc. II. 5.) shew that  $AB^2 = 4 \cdot CD^2 + 4 \cdot AD \cdot DB$ .

9. Divide a straight line into two parts, such that the sum of their squares may be the least possible.

10. Divide a line into two parts, such that the sum of their squares shall be double the square on another line.

11. Shew that the difference between the squares on the two unequal parts (fig. Euc. II. 9.) is equal to twice the rectangle contained by the whole line, and the part between the points of section.

12. Shew how in all the possible cases, a straight line may be *geometrically* divided into two such parts, that the sum of their squares shall be equal to a given square.

13. Divide a given straight line into two parts, such that the squares on the whole line and on one of the parts shall be equal to twice the square on the other part.

14. Any rectangle is the half of the rectangle contained by the diameters of the squares on its two sides.

15. If a straight line be divided into two equal and into two unequal parts, the squares on the two unequal parts are equal to twice the rectangle contained by the two unequal parts, together with four times the square on the line between the points of section.

16. If the points  $C, D$  be equidistant from the extremities of the straight line  $AB$ , shew that the squares constructed on  $AD$  and  $AC$ , exceed twice the rectangle  $AC, AD$  by the square constructed on  $CD$ .

17. If any point be taken in the plane of a parallelogram from which perpendiculars are let fall on the diagonal, and on the sides which include it, the rectangle of the diagonal and the perpendicular

on it, is equal to the sum or difference of the rectangles of the sides and the perpendiculars on them.

18.  $ABCD$  is a rectangular parallelogram, of which  $A, C$  are opposite angles,  $E$  any point in  $BC$ ,  $F$  any point in  $CD$ . Prove that twice the area of the triangle  $AEF$  together with the rectangle  $BE, DF$  is equal to the parallelogram  $AC$ .

## II.

19. Shew how to produce a given line, so that the rectangle contained by the whole line thus produced, and the produced part, shall be equal to the square (1) on the given line (2) on the part produced.

20. If in the figure Euc. II. 11, we join  $BF$  and  $CH$ , and produce  $CH$  to meet  $BF$  in  $L$ ,  $CL$  is perpendicular to  $BF$ .

21. If a line be divided, as in Euc. II. 11, the squares on the whole line and one of the parts are together three times the square on the other part.

22. If in the fig. Euc. II. 11, the points  $F, D$  be joined cutting  $AHB, GHK$  in  $f, d$  respectively; then shall  $Ff = Dd$ .

## III.

23. If from the three angles of a triangle, lines be drawn to the points of bisection of the opposite sides, the squares on the distances between the angles and the common intersection, are together one-third of the squares on the sides of the triangle.

24.  $ABC$  is a triangle of which the angle at  $C$  is obtuse, and the angle at  $B$  is half a right angle:  $D$  is the middle point of  $AB$ , and  $CE$  is drawn perpendicular to  $AB$ . Shew that the square on  $AC$  is double of the squares on  $AD$  and  $DE$ .

25. If an angle of a triangle be two-thirds of two right angles, shew that the square on the side subtending that angle is equal to the squares on the sides containing it, together with the rectangle contained by those sides.

26. The square described on a straight line drawn from one of the angles at the base of a triangle to the middle point of the opposite side, is equal to the sum or difference of the square on half the side bisected, and the rectangle contained between the base and that part of it, or of it produced, which is intercepted between the same angle and a perpendicular drawn from the vertex.

27.  $ABC$  is a triangle of which the angle at  $C$  is obtuse, and the angle at  $B$  is half a right angle:  $D$  is the middle point of  $AB$ , and  $CE$  is drawn perpendicular to  $AB$ . Shew that the square on  $AC$  is double of the squares on  $AD$  and  $DE$ .

28. Produce one side of a scalene triangle, so that the rectangle under it and the produced part may be equal to the difference of the squares on the other two sides.

29. Given the base of any triangle, the area, and the line bisecting the base, construct the triangle.

## IV.

30. Shew that the square on the hypotenuse of a right-angled triangle, is equal to four times the area of the triangle together with the square on the difference of the sides.

31. In the triangle  $ABC$ , if  $AD$  be the perpendicular let fall upon the side  $BC$ ; then the square on  $AC$  together with the rectangle contained by  $BC, BD$  is equal to the square on  $AB$  together with the rectangle  $CB, CD$ .

32.  $ABC$  is a triangle, right angled at  $C$ , and  $CD$  is the perpendicular let fall from  $C$  upon  $AB$ ; if  $HK$  is equal to the sum of the sides  $AC, CB$ , and  $LM$  to the sum of  $AB, CD$ , shew that the square on  $HK$  together with the square on  $CD$  is equal to the square on  $LM$ .

33.  $ABC$  is a triangle having the angle at  $B$  a right angle: it is required to find in  $AB$  a point  $P$  such that the square on  $AC$  may exceed the squares on  $AP$  and  $PC$  by half the square on  $AB$ .

34. In a right-angled triangle, the square on that side which is the greater of the two sides containing the right angle, is equal to the rectangle by the sum and difference of the other sides.

35. The hypotenuse  $AB$  of a right-angled triangle  $ABC$  is trisected in the points  $D, E$ ; prove that if  $CD, CE$  be joined, the sum of the squares on the sides of the triangle  $CDE$  is equal to two-thirds of the square on  $AB$ .

36. From the hypotenuse of a right-angled triangle portions are cut off equal to the adjacent sides: shew that the square on the middle segment is equivalent to twice the rectangle under the extreme segments.

## V.

37. Prove that the square on any straight line drawn from the vertex of an isosceles triangle to the base, is less than the square on a side of the triangle by the rectangle contained by the segments of the base: and conversely.

38. If from one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side, the rectangle contained by that side and the segment of it intercepted between the perpendicular and base, is equal to the half of the square described upon the base.

39. If in an isosceles triangle a perpendicular be let fall from one of the equal angles to the opposite side, the square on the perpendicular is equal to the square on the line intercepted between the other equal angle and the perpendicular, together with twice the rectangle contained by the segments of that side.

40. The square on the base of an isosceles triangle whose vertical angle is a right angle, is equal to four times the area of the triangle.

41. Describe an isosceles obtuse-angled triangle, such that the square on the side subtending the obtuse angle may be three times the square on either of the sides containing the obtuse angle.

42. If  $AB$ , one of the sides of an isosceles triangle  $ABC$  be produced beyond the base to  $D$ , so that  $BD = AB$ , shew that

$$CD^2 = AB^2 + 2 \cdot BC^2.$$

43. If  $ABC$  be an isosceles triangle, and  $DE$  be drawn parallel to the base  $BC$ , and  $EB$  be joined; prove that  $BE^2 = BC \times DE + CE^2$ .

44. If  $ABC$  be an isosceles triangle of which the angles at  $B$  and  $C$  are each double of  $A$ ; then the square on  $AC$  is equal to the square on  $BC$  together with the rectangle contained by  $AC$  and  $BC$ .

## VI.

45. Shew that in a parallelogram the squares on the diagonals are equal to the sum of the squares on all the sides.

46. If  $ABCD$  be any rectangle,  $A$  and  $C$  being opposite angles, and  $O$  any point either within or without the rectangle:

$$OA^2 + OC^2 = OB^2 + OD^2.$$

47. In any quadrilateral figure, the sum of the squares on the diagonals together with four times the square on the line joining their middle points, is equal to the sum of the squares on all the sides.

48. In any trapezium, if the opposite sides be bisected, the sum of the squares on the other two sides, together with the squares on the diagonals, is equal to the sum of the squares on the bisected sides, together with four times the square on the line joining the points of bisection.

49. The squares on the diagonals of a trapezium are together double the squares on the two lines joining the bisections of the opposite sides.

50. In any trapezium two of whose sides are parallel, the squares on the diagonals are together equal to the squares on its two sides which are not parallel, and twice the rectangle contained by the sides which are parallel.

51. If the two sides of a trapezium be parallel, shew that its area is equal to that of a rectangle contained by its altitude and half the sum of the parallel sides.

52. If a trapezium have two sides parallel, and the other two equal, shew that the rectangle contained by the two parallel sides, together with the square on one of the other sides, will be equal to the square on the straight line joining two opposite angles of the trapezium.

53. If squares be described on the sides of any triangle and the angular points of the squares be joined; the sum of the squares on the sides of the hexagonal figure thus formed is equal to four times the sum of the squares on the sides of the triangle.

## VII.

54. Find the side of a square equal to a given equilateral triangle.

55. Find a square which shall be equal to the sum of two given rectilineal figures.

56. To divide a given straight line so that the rectangle under its segments may be equal to a given rectangle.

57. Construct a rectangle equal to a given square and having the difference of its sides equal to a given straight line.

58. Shew how to describe a rectangle equal to a given square, and having one of its sides equal to a given straight line.



## BOOK III.

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### DEFINITIONS.

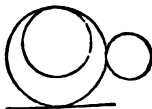
#### I.

**EQUAL** circles are those of which the diameters are equal, or from the centers of which the straight lines to the circumferences are equal.

This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centers coincide, the circles must likewise coincide, since the straight lines from the centers are equal.

#### II.

A straight line is said to **touch** a circle when it meets the circle, and being produced does not cut it.



#### III.

Circles are said to **touch** one another, which meet, but do not cut one another.

#### IV.

Straight lines are said to be **equally distant** from the center of a circle, when the perpendiculars drawn to them from the center are equal.



#### V.

And the straight line on which the greater perpendicular falls, is said to be **further** from the center.

#### VI.

A **segment** of a circle is the figure contained by a straight line, and the *arc* or the part of the circumference which it cuts off.



## VII.

The angle of a segment is that which is contained by a straight line and a part of the circumference.

## VIII.

An angle in a segment is any angle contained by two straight lines drawn from any point in the arc of the segment, to the extremities of the straight line which is the base of the segment.

## IX.

An angle is said to insist or stand upon the part of the circumference intercepted between the straight lines that contain the angle.



## X.

A sector of a circle is the figure contained by two straight lines drawn from the center and the arc between them.



## XI.

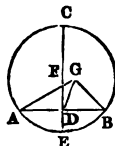
Similar segments of circles are those in which the angles are equal, or which contain equal angles.



## PROPOSITION I. PROBLEM.

*To find the center of a given circle.*

Let  $ABC$  be the given circle: it is required to find its center.



Draw within it any straight line  $AB$  to meet the circumference in  $A, B$ ; and bisect  $AB$  in  $D$ ; (I. 10.) from the point  $D$  draw  $DC$  at right angles to  $AB$ , (I. 11.) meeting the circumference in  $C$ , produce  $CD$  to  $E$  to meet the circumference again in  $E$ , and bisect  $CE$  in  $F$ .

Then the point  $F$  shall be the center of the circle  $ABC$ .

For, if it be not, if possible, let  $G$  be the center, and join  $GA, GD, GB$ .

Then, because  $DA$  is equal to  $DB$ , (constr.)

and  $DG$  common to the two triangles  $ADG, BDG$ ,

the two sides  $AD, DG$ , are equal to the two  $BD, DG$ , each to each;

and the base  $GA$  is equal to the base  $GB$ , (I. def. 15.)

because they are drawn from the center  $G$ :

therefore the angle  $ADG$  is equal to the angle  $GDB$ : (I. 8.)

but when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of the angles is a right angle; (I. def. 10.)

therefore the angle  $GDB$  is a right angle:

but  $FDB$  is likewise a right angle; (constr.)

wherefore the angle  $FDB$  is equal to the angle  $GDB$ , (ax. 1.)

the greater angle equal to the less, which is impossible;

therefore  $G$  is not the center of the circle  $ABC$ .

In the same manner it can be shewn that no other point out of the line  $CE$  is the center;

and since  $CE$  is bisected in  $F$ ,

any other point in  $CE$  divides  $CE$  into unequal parts, and cannot be the center.

Therefore no point but  $F$  is the center of the circle  $ABC$ .

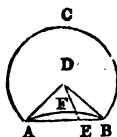
Which was to be found.

COR. From this it is manifest, that if in a circle a straight line bisects another at right angles, the center of the circle is in the line which bisects the other.

## PROPOSITION II. THEOREM.

*If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.*

Let  $ABC$  be a circle, and  $A, B$  any two points in the circumference. Then the straight line drawn from  $A$  to  $B$  shall fall within the circle.



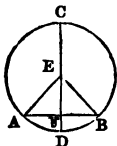
For if  $AB$  do not fall within the circle,  
 let it fall, if possible, without the circle as  $AEB$ ;  
 find  $D$  the center of the circle  $ABC$ , (III. 1.) and join  $DA$ ,  $DB$ ;  
 in the circumference  $AB$  take any point  $F$ ,  
 join  $DF$ , and produce it to meet  $AB$  in  $E$ .  
 Then, because  $DA$  is equal to  $DB$ , (I. def. 15.)  
 therefore the angle  $DBA$  is equal to the angle  $DAB$ ; (I. 5.)  
 and because  $AE$ , a side of the triangle  $DAE$ , is produced to  $B$ ,  
 the exterior angle  $DEB$  is greater than the interior and opposite  
 angle  $DAE$ ; (I. 16.)  
 but  $DAE$  was proved to be equal to the angle  $DBE$ ;  
 therefore the angle  $DEB$  is greater than the angle  $DBE$ ;  
 but to the greater angle the greater side is opposite, (I. 19.)  
 therefore  $DB$  is greater than  $DE$ ;  
 but  $DB$  is equal to  $DF$ ; (I. def. 15.)  
 wherefore  $DF$  is greater than  $DE$ ,  
 the less than the greater, which is impossible;  
 therefore the straight line drawn from  $A$  to  $B$  does not fall without  
 the circle.  
 In the same manner, it may be demonstrated that it does not fall  
 upon the circumference;  
 therefore it falls within it.  
 Wherefore, if any two points, &c. Q.E.D.

## PROPOSITION III. THEOREM.

*If a straight line drawn through the center of a circle bisect a straight line in it which does not pass through the center, it shall cut it at right angles: and conversely, if it cut it at right angles, it shall bisect it.*

Let  $ABC$  be a circle; and let  $CD$ , a straight line drawn through the center, bisect any straight line  $AB$ , which does not pass through the center, in the point  $F$ .

Then  $CD$  shall cut  $AB$  at right angles.



Take  $E$  the center of the circle, (III. 1.) and join  $EA$ ,  $EB$ .

Then, because  $AF$  is equal to  $FB$ , (hyp.)  
 and  $FE$  common to the two triangles  $AFE$ ,  $BFE$ ,

there are two sides in the one equal to two sides in the other, each to each;

and the base  $EA$  is equal to the base  $EB$ ; (I. def. 15.)

therefore the angle  $AFE$  is equal to the angle  $BFE$ ; (I. 8.)

but when a straight line standing upon another straight line makes the adjacent angles equal to one another,

each of them is a right angle; (I. def. 10.)

therefore each of the angles  $AFE$ ,  $BFE$ , is a right angle:

wherefore the straight line  $CD$ , drawn through the center, bisecting another  $AB$  that does not pass through the center, cuts the same at right angles.

Conversely, let  $CD$  cut  $AB$  at right angles.

Then  $CD$  shall also bisect  $AB$ , that is,  $AF$  shall be equal to  $FB$ .

The same construction being made,

because,  $EB$ ,  $EA$ , from the center are equal to one another, (I. def. 15.)

therefore the angle  $EAF$  is equal to the angle  $EBF$ ; (I. 5.)

and the right angle  $AFE$  is equal to the right angle  $BFE$ ; (I. def. 10.)

therefore, in the two triangles,  $EAF$ ,  $EBF$ ,

there are two angles in the one equal to two angles in the other, each to each;

and the side  $EF$ , which is opposite to one of the equal angles in each, is common to both;

therefore the other sides are equal; (I. 26.)

therefore  $AF$  is equal to  $FB$ .

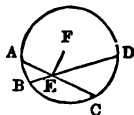
Wherefore, if a straight line, &c. Q. E. D.

#### PROPOSITION IV. THEOREM.

*If in a circle two straight lines cut one another, which do not both pass through the center, they do not bisect each other.*

Let  $ABCD$  be a circle, and  $AC$ ,  $BD$  two straight lines in it which cut one another in the point  $E$ , and do not both pass through the center.

Then  $AC$ ,  $BD$ , shall not bisect one another.



For, if it be possible, let  $AE$  be equal to  $EC$ , and  $BE$  to  $ED$ .

If one of the lines pass through the center, it is plain that it cannot be bisected by the other which does not pass through the center:

but if neither of them pass through the center,

find  $F$  the center of the circle, (III. 1.) and join  $EF$ .

Then because  $FE$ , a straight line drawn through the center, bisects another  $AC$  which does not pass through the center, (hyp.)

therefore  $FE$  cuts  $AC$  at right angles: (III. 3.)

wherefore  $FEA$  is a right angle.

Again, because the straight line  $FE$  bisects the straight line  $BD$ , which does not pass through the center, (hyp.)

therefore  $FE$  cuts  $BD$  at right angles: (III. 3.)

wherefore  $FEB$  is a right angle:

but  $FEA$  was shewn to be a right angle;

therefore the angle  $FEA$  is equal to the angle  $FEB$ , (ax. 1.)

the less equal to the greater, which is impossible:

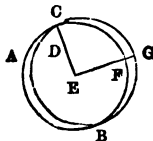
therefore  $AC$ ,  $BD$  do not bisect one another.

Wherefore, if in a circle, &c. Q. E. D.

### PROPOSITION V. THEOREM.

*If two circles cut one another, they shall not have the same center.*

Let the two circles  $ABC$ ,  $CDG$ , cut one another in the points  $B$ ,  $C$ .  
They shall not have the same center.



If possible, let  $E$  be the center of the two circles; join  $EC$ , and draw any straight line  $EFG$  meeting the circumferences in  $F$  and  $G$ .

And because  $E$  is the center of the circle  $ABC$ ,

therefore  $EF$  is equal to  $EC$ : (1. def. 15.)

again, because  $E$  is the center of the circle  $CDG$ ,

therefore  $EG$  is equal to  $EC$ : (1. def. 15.)

but  $EF$  was shewn to be equal to  $EC$ ;

therefore  $EF$  is equal to  $EG$ , (ax. 1.)

the less line equal to the greater, which is impossible.

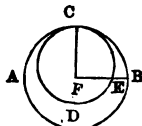
Therefore  $E$  is not the center of the circles  $ABC$ ,  $CDG$ .

Wherefore, if two circles, &c. Q. E. D.

### PROPOSITION VI. THEOREM.

*If one circle touch another internally, they shall not have the same center.*

Let the circle  $CDE$  touch the circle  $ABC$  internally in the point  $C$ .  
They shall not have the same center.



If possible, let  $F$  be the center of the two circles: join  $FC$ , and draw any straight line  $FEB$ , meeting the circumferences in  $E$  and  $B$ .

And because  $F$  is the center of the circle  $ABC$ ,

$FB$  is equal to  $FC$ ; (1. def. 15.)

also, because  $F$  is the center of the circle  $CDE$ ,  
 $FE$  is equal to  $FC$ : (1. def. 15.)  
 but  $FB$  was shewn to be equal to  $FC$ ;  
 therefore  $FE$  is equal to  $FB$ , (ax. 1.)  
 the less line equal to the greater, which is impossible:  
 therefore  $F$  is not the center of the circles  $ABC$ ,  $CDE$ .  
 Therefore, if two circles, &c. Q.E.D.

### PROPOSITION VII. THEOREM.

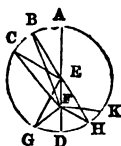
*If any point be taken in the diameter of a circle which is not the center, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the center is, and the other part of that diameter is the least; and, of the rest, that which is nearer to the line which passes through the center is always greater than one more remote: and from the same point there can be drawn only two equal straight lines to the circumference one upon each side of the diameter.*

Let  $ABCD$  be a circle, and  $AD$  its diameter, in which let any point  $F$  be taken which is not the center:

let the center be  $E$ .

Then, of all the straight lines  $FB$ ,  $FC$ ,  $FG$  &c. that can be drawn from  $F$  to the circumference,

$FA$ , that in which the center is, shall be the greatest,  
 and  $FD$ , the other part of the diameter  $AD$ , shall be the least:  
 and of the rest,  $FB$ , the nearer to  $FA$ , shall be greater than  $FC$   
 the more remote, and  $FC$  greater than  $FG$ .



Join  $BE$ ,  $CE$ ,  $GE$ .

Because two sides of a triangle are greater than the third side, (1. 20.)

therefore  $BE$ ,  $EF$  are greater than  $BF$ :

but  $AE$  is equal to  $BE$ ; (1. def. 15.)

therefore  $AE$ ,  $EF$ , that is,  $AF$  is greater than  $BF$ .

Again, because  $BE$  is equal to  $CE$ ,

and  $FE$  common to the triangles  $BEF$ ,  $CEF$ ,

the two sides  $BE$ ,  $EF$  are equal to the two  $CE$ ,  $EF$ , each to each;

but the angle  $BEF$  is greater than the angle  $CEF$ ; (ax. 9.)

therefore the base  $BF$  is greater than the base  $CF$ . (1. 24.)

For the same reason  $CF$  is greater than  $GF$ .

Again, because  $GF$ ,  $FE$  are greater than  $EG$ , (1. 20.)

and  $EG$  is equal to  $ED$ ;

therefore  $GF$ ,  $FE$  are greater than  $ED$ :

take away the common part  $FE$ ,

and the remainder  $GF$  is greater than the remainder  $FD$ . (ax. 5.)

Therefore,  $FA$  is the greatest,  
and  $FD$  the least of all the straight lines from  $F$  to the circumference;  
and  $BF$  is greater than  $CF$ , and  $CF$  than  $GF$ .

Also, there can be drawn only two equal straight lines from the point  $F$  to the circumference, one upon each side of the diameter.

At the point  $E$ , in the straight line  $EF$ , make the angle  $FEH$  equal to the angle  $FEG$ , (I. 23.) and join  $FH$ .

Then, because  $GE$  is equal to  $EH$ , (I. def. 15.)

and  $EF$  common to the two triangles  $GEF$ ,  $HEF$ ;

the two sides  $GE$ ,  $EF$  are equal to the two  $HE$ ,  $EF$ , each to each;

and the angle  $GEF$  is equal to the angle  $HEF$ ; (constr.)

therefore the base  $FG$  is equal to the base  $FH$ : (I. 4.)

but, besides  $FH$ , no other straight line can be drawn from  $F$  to the circumference equal to  $FG$ :

for, if possible, let it be  $FK$ :

and because  $FK$  is equal to  $FG$ , and  $FG$  to  $FH$ ,

therefore  $FK$  is equal to  $FH$ ; (ax. 1.)

that is, a line nearer to that which passes through the center, is equal to one which is more remote;

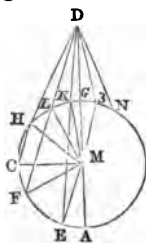
which has been proved to be impossible.

Therefore, if any point be taken, &c. Q.E.D.

#### PROPOSITION VIII. THEOREM.

*If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the center; of those which fall upon the concave part of the circumference, the greatest is that which passes through the center; and of the rest, that which is nearer to the one passing through the center is always greater than one more remote: but of those which fall upon the convex part of the circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote; and only two equal straight lines can be drawn from the same point to the circumference, one upon each side of the line which passes through the center.*

Let  $ABC$  be a circle, and  $D$  any point without it, from which let the straight lines  $DA$ ,  $DE$ ,  $DF$ ,  $DC$  be drawn to the circumference, whereof  $DA$  passes through the center.



Of those which fall upon the concave part of the circumference  $AECF$ , the greatest shall be  $DA$ , which passes through the center;



and any line nearer to it shall be greater than one more remote,  
viz.  $DE$  shall be greater than  $DF$ , and  $DF$  greater than  $DC$ ;  
but of those which fall upon the convex part of the circumference  $HLKG$ ,  
the least shall be  $DG$  between the point  $D$  and the diameter  $AG$ ;

and any line nearer to it shall be less than one more remote,  
viz.  $DK$  less than  $DL$ , and  $DL$  less than  $DH$ .

Take  $M$  the center of the circle  $ABC$ , (III. 1.)

and join  $ME$ ,  $MF$ ,  $MC$ ,  $MK$ ,  $ML$ ,  $MH$ .

And because  $AM$  is equal to  $ME$ ,

add  $MD$  to each of these equals,

therefore  $AD$  is equal to  $EM$ ,  $MD$ : (ax. 2.)

but  $EM$ ,  $MD$  are greater than  $ED$ ; (I. 20.)

therefore also  $AD$  is greater than  $ED$ .

Again, because  $ME$  is equal to  $MF$ , and  $MD$  common to the triangles  $EMD$ ,  $FMD$ ;  $EM$ ,  $MD$ , are equal to  $FM$ ,  $MD$ , each to each;

but the angle  $EMD$  is greater than the angle  $FMD$ ; (ax. 9.)

therefore the base  $ED$  is greater than the base  $FD$ . (I. 24.)

In like manner it may be shewn that  $FD$  is greater than  $CD$ .

Therefore  $DA$  is the greatest;

and  $DE$  greater than  $DF$ , and  $DF$  greater than  $DC$ .

And, because  $MK$ ,  $KD$  are greater than  $MD$ , (I. 20.)

and  $MK$  is equal to  $MG$ , (I. def. 15.)

the remainder  $KD$  is greater than the remainder  $GD$ , (ax. 5.)

that is,  $GD$  is less than  $KD$ :

and because  $MLD$  is a triangle, and from the points  $M$ ,  $D$ , the extremities of its side  $MD$ , the straight lines  $MK$ ,  $DK$  are drawn to the point  $K$  within the triangle,

therefore  $MK$ ,  $KD$  are less than  $ML$ ,  $LD$ : (I. 21.)

but  $MK$  is equal to  $ML$ ; (I. def. 15.)

therefore, the remainder  $DK$  is less than the remainder  $DL$ . (ax. 5.)

In like manner it may be shewn, that  $DL$  is less than  $DH$ .

Therefore,  $DG$  is the least, and  $DK$  less than  $DL$ , and  $DL$  less than  $DH$ .

Also, there can be drawn only two equal straight lines from the point  $D$  to the circumference, one upon each side of the line which passes through the center.

At the point  $M$ , in the straight line  $MD$ ,

make the angle  $DMB$  equal to the angle  $DMK$ , (I. 23.) and join  $DB$ .

And because  $MK$  is equal to  $MB$ , and  $MD$  common to the triangles  $KMD$ ,  $BMD$ ,

the two sides  $KM$ ,  $MD$  are equal to the two  $BM$ ,  $MD$ , each to each;

and the angle  $KMD$  is equal to the angle  $BMD$ ; (constr.)

therefore the base  $DK$  is equal to the base  $DB$ : (I. 4.)

but, besides  $DB$ , no straight line equal to  $DK$  can be drawn from  $D$  to the circumference,

for, if possible, let it be  $DN$ ;

and because  $DK$  is equal to  $DN$ , and also to  $DB$ ,

therefore  $DB$  is equal to  $DN$ ;

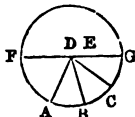
that is, a line nearer to the least is equal to one more remote,  
which has been proved to be impossible.

If therefore, any point, &c. Q. E. D.

## PROPOSITION IX. THEOREM.

*If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the center of the circle.*

Let the point  $D$  be taken within the circle  $ABC$ , from which to the circumference there fall more than two equal straight lines, viz.  $DA, DB, DC$ .  
Then the point  $D$  shall be the center of the circle.



For, if not, let  $E$ , if possible, be the center:  
join  $DE$ , and produce it to meet the circumference in  $F, G$ ;  
then  $FG$  is a diameter of the circle  $ABC$ : (I. def. 17.)  
and because in  $FG$ , the diameter of the circle  $ABC$ , there is taken the point  $D$ , which is not the center,  
therefore  $DG$  is the greatest line drawn from it to the circumference,  
and  $DC$  is greater than  $DB$ , and  $DB$  greater than  $DA$ : (III. 7.)  
but these lines are likewise equal, (hyp.) which is impossible:  
therefore  $E$  is not the center of the circle  $ABC$ .

In like manner it may be demonstrated,  
that no other point but  $D$  is the center;

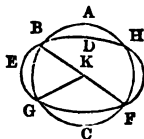
$D$  therefore is the center.

Wherefore, if a point be taken, &c. Q. E. D.

## PROPOSITION X. THEOREM.

*One circumference of a circle cannot cut another in more than two points.*

If it be possible, let the circumference  $ABC$  cut the circumference  $DEF$  in more than two points, viz. in  $B, G, F$ .



Take the center  $K$  of the circle  $ABC$ , (III. 1.) and join  $KB, KG, KF$ .

Then because  $K$  is the center of the circle  $ABC$ ,  
therefore  $KB, KG, KF$  are all equal to each other: (I. def. 15.)  
and because within the circle  $DEF$  there is taken the point  $K$ , from which to the circumference  $DEF$  fall more than two equal straight lines  $KB, KG, KF$ ;

therefore the point  $K$  is the center of the circle  $DEF$ : (III. 9.)

but  $K$  is also the center of the circle  $ABC$ ; (constr.)

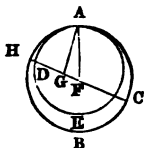
therefore the same point is the center of two circles that cut one another, which is impossible. (III. 5.)

Therefore, one circumference of a circle cannot cut another in more than two points. Q. E. D.

### PROPOSITION XI. THEOREM.

*If one circle touch another internally in any point, the straight line which joins their centers being produced, shall pass through that point of contact.*

Let the circle  $ADE$  touch the circle  $ABC$  internally in the point  $A$ ; and let  $F$  be the center of the circle  $ABC$ , and  $G$  the center of the circle  $ADE$ ; then the straight line which joins the centers  $F, G$ , being produced, shall pass through the point  $A$ .



For, if  $FG$  produced do not pass through the point  $A$ , let it fall otherwise, if possible, as  $FGDH$ , and join  $AF, AG$ .

Then, because two sides of a triangle are together greater than the third side, (I. 20.)

therefore  $FG, GA$  are greater than  $FA$ :

but  $FA$  is equal to  $FH$ ; (I. def. 15.)

therefore  $FG, GA$  are greater than  $FH$ :

take away from these unequals the common part  $FG$ ;

therefore the remainder  $AG$  is greater than the remainder  $GH$ ; (ax. 5.)

but  $AG$  is equal to  $GD$ ; (I. def. 15.)

therefore  $GD$  is greater than  $GH$ ,

the less than the greater, which is impossible.

Therefore the straight line which joins the points  $F, G$ , being produced, cannot fall otherwise than upon the point  $A$ ,

that is, it must pass through it.

Therefore, if one circle, &c. Q. E. D.

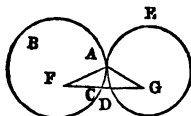
### PROPOSITION XII. THEOREM.

*If two circles touch each other externally in any point, the straight line which joins their centers, shall pass through that point of contact.*

Let the two circles  $ABC, ADE$ , touch each other externally in the point  $A$ ;

and let  $F$  be the center of the circle  $ABC$ , and  $G$  the center of  $ADE$ .

Then the straight line which joins the points  $F, G$ , shall pass through the point of contact  $A$ .



If not, let it pass otherwise, if possible, as  $FCDG$ , and join  $FA$ ,  $AG$ .

And because  $F$  is the center of the circle  $ABC$ ,

$FA$  is equal to  $FC$ :

also, because  $G$  is the center of the circle  $ADE$ ,

$GA$  is equal to  $GD$ :

therefore  $FA$ ,  $AG$  are equal to  $FC$ ,  $DG$ ; (ax. 2.)

wherefore the whole  $FG$  is greater than  $FA$ ,  $AG$ :

but  $FG$  is less than  $FA$ ,  $AG$ ; (I. 20.) which is impossible:

therefore the straight line which joins the points  $F$ ,  $G$ , cannot pass otherwise than through  $A$  the point of contact,

that is,  $FG$  must pass through the point  $A$ .

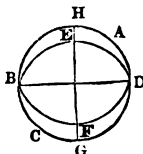
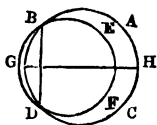
Therefore, if two circles, &c. Q. E. D.

### PROPOSITION XIII. THEOREM.

*One circle cannot touch another in more points than in one, whether it touches it on the inside or outside.*

For, if it be possible, let the circle  $EBF$  touch the circle  $ABC$  in more points than in one.

and first on the inside, in the points  $B$ ,  $D$ .



Join  $BD$ , and draw  $GH$  bisecting  $BD$  at right angles. (I. 11.)

Because the points  $B$ ,  $D$  are in the circumferences of each of the circles,

therefore the straight line  $BD$  falls within each of them; (III. 2.)

therefore their centers are in the straight line  $GH$  which bisects  $BD$  at right angles; (III. 1. Cor.)

therefore  $GH$  passes through the point of contact: (III. 11.)

but it does not pass through it,

because the points  $B$ ,  $D$  are without the straight line  $GH$ ;

which is absurd:

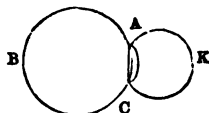
therefore one circle cannot touch another on the inside in more points than in one.

Nor can two circles touch one another on the outside in more than in one point.

For, if it be possible,

let the circle  $ACK$  touch the circle  $ABC$  in the points  $A$ ,  $C$ ;

join  $AC$ .



Because the two points  $A, C$  are in the circumference of the circle  $ACK$ ,  
therefore the straight line  $AC$  which joins them, falls within the circle  
 $ACK$ : (III. 2.)

but the circle  $ACK$  is without the circle  $ABC$ ; (hyp.)

therefore the straight line  $AC$  is without this last circle:

but, because the points  $A, C$  are in the circumference of the circle  $ABC$ ,  
the straight line  $AC$  must be within the same circle, (III. 2.)  
which is absurd;

therefore one circle cannot touch another on the outside in more than  
in one point:

and it has been shewn, that they cannot touch on the inside in more  
points than in one.

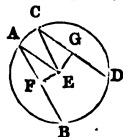
Therefore, one circle, &c. Q.E.D.

#### PROPOSITION XIV. THEOREM.

*Equal straight lines in a circle are equally distant from the center;  
and conversely, those which are equally distant from the center, are equal  
to one another.*

Let the straight lines  $AB, CD$ , in the circle  $ABDC$ , be equal to  
one another.

Then  $AB$  and  $CD$  shall be equally distant from the center.



Take  $E$  the center of the circle  $ABDC$ , (III. 1.)  
from  $E$  draw  $EF, EG$  perpendiculars to  $AB, CD$ , (I. 12.) and join  
 $EA, EC$ .

Then, because the straight line  $EF$  passing through the center,  
cuts  $AB$ , which does not pass through the center, at right angles;

$EF$  bisects  $AB$  in the point  $F$ : (III. 3.)

therefore  $AF$  is equal to  $FB$ , and  $AB$  double of  $AF$ .

For the same reason  $CD$  is double of  $CG$ :

but  $AB$  is equal to  $CD$ : (hyp.)

therefore  $AF$  is equal to  $CG$ . (ax. 7.)

And because  $AE$  is equal to  $EC$ , (I. def. 15.)

the square on  $AE$  is equal to the square on  $EC$ :

but the squares on  $AF, FE$  are equal to the square on  $AE$ , (I. 47.)

because the angle  $AFE$  is a right angle;

and for the same reason, the squares on  $EG$ ,  $GC$  are equal to the square on  $EC$ ;

therefore the squares on  $AF$ ,  $FE$  are equal to the squares on  $CG$ ,  $GE$ : (ax. 1.)

but the square on  $AF$  is equal to the square on  $CG$ ,

because  $AF$  is equal to  $CG$ ;

therefore the remaining square on  $EF$  is equal to the remaining square on  $EG$ , (ax. 3.)

and the straight line  $EF$  is therefore equal to  $EG$ :

but straight lines in a circle are said to be equally distant from the center, when the perpendiculars drawn to them from the center are equal; (III. def. 4.)

therefore  $AB$ ,  $CD$  are equally distant from the center.

Conversely, let the straight lines  $AB$ ,  $CD$  be equally distant from the center, (III. def. 4.)

that is, let  $FE$  be equal to  $EG$ ;

then  $AB$  shall be equal to  $CD$ .

For the same construction being made,

it may, as before, be demonstrated,

that  $AB$  is double of  $AF$ , and  $CD$  double of  $CG$ ,

and that the squares on  $FE$ ,  $AF$  are equal to the squares on  $EG$ ,  $GC$ :

but the square on  $FE$  is equal to the square on  $EG$ ,

because  $FE$  is equal to  $EG$ ; (hyp.)

therefore the remaining square on  $AF$  is equal to the remaining square on  $CG$ : (ax. 3.)

and the straight line  $AF$  is therefore equal to  $CG$ :

but  $AB$  was shewn to be double of  $AF$ , and  $CD$  double of  $CG$ ;

wherefore  $AB$  is equal to  $CD$ . (ax. 6.)

Therefore equal straight lines, &c. Q. E. D.

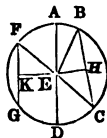
#### PROPOSITION XV. THEOREM.

*The diameter is the greatest straight line in a circle; and of the rest, that which is nearer to the center is always greater than one more remote: and conversely the greater is nearer to the center than the less.*

Let  $ABCD$  be a circle, of which the diameter is  $AD$ , and the center  $E$ ;

and let  $BC$  be nearer to the center than  $FG$ .

Then  $AD$  shall be greater than any straight line  $BC$ , which is not a diameter, and  $BC$  shall be greater than  $FG$ .



From  $E$  draw  $EH$ , perpendicular to  $BC$ , and  $EK$  to  $FG$ , (I. 12.) and join  $EB$ ,  $EC$ ,  $EF$ .

And because  $AE$  is equal to  $EB$ , and  $ED$  to  $EC$ , (I. def. 15.)

therefore  $AD$  is equal to  $EB$ ,  $EC$ : (ax. 2.)

but  $EB$ ,  $EC$  are greater than  $BC$ ; (I. 20.)

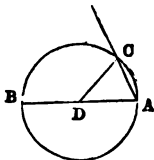
wherefore also  $AD$  is greater than  $BC$ .

And, because  $BC$  is nearer to the center than  $FG$ , (hyp.)  
 therefore  $EH$  is less than  $EK$ : (III. def. 5.)  
 but, as was demonstrated in the preceding proposition,  
 $BC$  is double of  $BH$ , and  $FG$  double of  $FK$ ,  
 and the squares on  $EH$ ,  $HB$  are equal to the squares on  $EK$ ,  $KF$ :  
 but the square on  $EH$  is less than the square on  $EK$ ,  
 because  $EH$  is less than  $EK$ ;  
 therefore the square on  $BH$  is greater than the square on  $FK$ ,  
 and the straight line  $BH$  greater than  $FK$ ,  
 and therefore  $BC$  is greater than  $FG$ .  
 Next, let  $BC$  be greater than  $FG$ ;  
 then  $BC$  shall be nearer to the center than  $FG$ , that is, the same construction being made,  $EH$  shall be less than  $EK$ . (III. def. 5.)  
 Because  $BC$  is greater than  $FG$ ,  
 $BH$  likewise is greater than  $FK$ :  
 and the squares on  $BH$ ,  $HE$  are equal to the squares on  $FK$ ,  $KE$ ,  
 of which the square on  $BH$  is greater than the square on  $FK$ ,  
 because  $BH$  is greater than  $FK$ :  
 therefore the square on  $EH$  is less than the square on  $EK$ ,  
 and the straight line  $EH$  less than  $EK$ :  
 and therefore  $BC$  is nearer to the center than  $FG$ . (III. def. 5.)  
 Wherefore the diameter, &c. Q. E. D.

### PROPOSITION XVI. THEOREM.

*The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn from the extremity between that straight line and the circumference, so as not to cut the circle: or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line which is at right angles to it, as not to cut the circle.*

Let  $ABC$  be a circle, the center of which is  $D$ , and the diameter  $AB$ .  
 Then the straight line drawn at right angles to  $AB$  from its extremity  $A$ , shall fall without the circle.



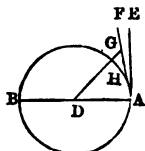
For, if it does not, let it fall, if possible, within the circle, as  $AC$ ;  
 and draw  $DC$  to the point  $C$ , where it meets the circumference.

And because  $DA$  is equal to  $DC$ , (I. def. 15.)  
 the angle  $DAC$  is equal to the angle  $ACD$ : (I. 5.)  
 but  $DAC$  is a right angle; (hyp.)  
 therefore  $ACD$  is a right angle;  
 and therefore the angles  $DAC$ ,  $ACD$  are equal to two right angles;  
 which is impossible: (I. 17.)  
 therefore the straight line drawn from  $A$  at right angles to  $BA$ , does not fall within the circle.

In the same manner it may be demonstrated,  
that it does not fall upon the circumference;  
therefore it must fall without the circle, as  $AE$ .

Also, between the straight line  $AE$  and the circumference, no straight line can be drawn from the point  $A$  which does not cut the circle.

For, if possible, let  $AF$  fall between them,



and from the point  $D$ , let  $DG$  be drawn perpendicular to  $AF$ , (I. 12.)  
and let it meet the circumference in  $H$ .

And because  $AGD$  is a right angle,  
and  $DAG$  less than a right angle, (I. 17.)  
therefore  $DA$  is greater than  $DG$ : (I. 19.)  
but  $DA$  is equal to  $DH$ ; (I. def. 15.)

therefore  $DH$  is greater than  $DG$ ,  
the less than the greater, which is impossible:

therefore no straight line can be drawn from the point  $A$ , between  
 $AE$  and the circumference, which does not cut the circle:

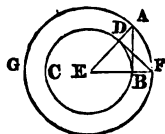
or, which amounts to the same thing, however great an acute angle  
a straight line makes with the diameter at the point  $A$ , or however  
small an angle it makes with  $AE$ , the circumference must pass be-  
tween that straight line and the perpendicular  $AE$ . Q.E.D.

COR. From this it is manifest, that the straight line which is  
drawn at right angles to the diameter of a circle from the extremity  
of it touches the circle; (III. def. 2.) and that it touches it only in one  
point, because, if it did meet the circle in two, it would fall within it.  
(III. 2.) "Also, it is evident, that there can be but one straight line  
which touches the circle in the same point."

#### PROPOSITION XVII. PROBLEM.

*To draw a straight line from a given point, either without or in the cir-  
cumference, which shall touch a given circle.*

First, let  $A$  be a given point without the given circle  $BCD$ ;  
it is required to draw a straight line from  $A$  which shall touch the circle.



Find the center  $E$  of the circle, (III. 1.) and join  $AE$ ;  
and from the center  $E$ , at the distance  $EA$ , describe the circle  $AFG$ ;  
from the point  $D$  draw  $DF$  at right angles to  $EA$ , (I. 11.) meeting  
the circumference of the circle  $AFG$  in  $F$ ;  
and join  $EBF$ ,  $AB$ .

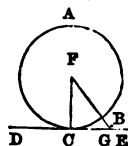


Then  $AB$  shall touch the circle  $BCD$  in the point  $B$ .  
 Because  $E$  is the center of the circles  $BCD$ ,  $AFG$ , (I. def. 15.)  
 therefore  $EA$  is equal to  $EF$ , and  $ED$  to  $EB$ ;  
 therefore the two sides  $AE$ ,  $EB$ , are equal to the two  $FE$ ,  $ED$ ,  
 each to each:  
 and they contain the angle at  $E$  common to the two triangles  $AEB$ ,  
 $FED$ ;  
 therefore the base  $DF$  is equal to the base  $AB$ , (I. 4.)  
 and the triangle  $FED$  to the triangle  $AEB$ ,  
 and the other angles to the other angles:  
 therefore the angle  $EBA$  is equal to the angle  $EDF$ :  
 but  $EDF$  is a right angle, (constr.)  
 wherefore  $EBA$  is a right angle: (ax. 1.)  
 and  $EB$  is drawn from the center:  
 but a straight line drawn from the extremity of a diameter, at right  
 angles to it, touches the circle: (III. 16. Cor.)  
 therefore  $AB$  touches the circle;  
 and it is drawn from the given point  $A$ .  
 Secondly, if the given point be in the circumference of the circle,  
 as the point  $D$ ,  
 draw  $DE$  to the center  $E$ , and  $DF$  at right angles to  $DE$ :  
 then  $DF$  touches the circle. (III. 16. Cor.) Q. E. F.

### PROPOSITION XVIII. THEOREM.

*If a straight line touch a circle, the straight line drawn from the center to the point of contact, shall be perpendicular to the line touching the circle.*

Let the straight line  $DE$  touch the circle  $ABC$  in the point  $C$ ;  
 take the center  $F$ , and draw the straight line  $FC$ . (III. 1.)  
 Then  $FC$  shall be perpendicular to  $DE$ .



If  $FC$  be not perpendicular to  $DE$ ; from the point  $F$ , if possible,  
 let  $FBG$  be drawn perpendicular to  $DE$ .

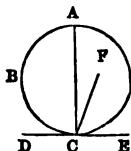
And because  $FGC$  is a right angle,  
 therefore  $GCF$  is an acute angle; (I. 17.)  
 and to the greater angle the greater side is opposite: (I. 19.)  
 therefore  $FC$  is greater than  $FG$ :  
 but  $FC$  is equal to  $FB$ ; (I. def. 15.)  
 therefore  $FB$  is greater than  $FG$ ,  
 the less than the greater, which is impossible:  
 therefore  $FG$  is not perpendicular to  $DE$ .

In the same manner it may be shewn,  
 that no other line is perpendicular to  $DE$  besides  $FC$ ,  
 that is,  $FC$  is perpendicular to  $DE$ .  
 Therefore, if a straight line, &c. Q. E. D.

## PROPOSITION XIX. THEOREM.

*If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the center of the circle shall be in that line.*

Let the straight line  $DE$  touch the circle  $ABC$  in  $C$ ,  
and from  $C$  let  $CA$  be drawn at right angles to  $DE$ .  
Then the center of the circle shall be in  $CA$ .



For, if not, let  $F$  be the center, if possible, and join  $CF$ .

Because  $DE$  touches the circle  $ABC$ ,  
and  $FC$  is drawn from the center to the point of contact,  
therefore  $FC$  is perpendicular to  $DE$ ; (III. 18.)

therefore  $FCE$  is a right angle:

but  $ACE$  is also a right angle; (hyp.)

therefore the angle  $FCE$  is equal to the angle  $ACE$ , (ax. 1.)

the less to the greater, which is impossible:

therefore  $F$  is not the center of the circle  $ABC$ .

In the same manner it may be shewn,

that no other point which is not in  $CA$ , is the center;

that is, the center of the circle is in  $CA$ .

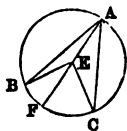
Therefore, if a straight line, &c. Q. E. D.

## PROPOSITION XX. THEOREM.

*The angle at the center of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.*

Let  $ABC$  be a circle, and  $BEC$  an angle at the center, and  $BAC$  an angle at the circumference, which have  $BC$  the same part of the circumference for their base.

Then the angle  $BEC$  shall be double of the angle  $BAC$ .



Join  $AE$ , and produce it to  $F$ .

First, let the center of the circle be within the angle  $BAC$ .

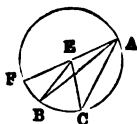
Because  $EA$  is equal to  $EB$ ,

therefore the angle  $EBA$  is equal to the angle  $EAB$ ; (I. 5.)

therefore the angles  $EAB$ ,  $EBA$  are double of the angle  $EAB$ :

but the angle  $BEF$  is equal to the angles  $EAB$ ,  $EBA$ ; (I. 32.)

therefore also the angle  $BEF$  is double of the angle  $EAB$ :  
 for the same reason, the angle  $FEC$  is double of the angle  $EAC$ :  
 therefore the whole angle  $BEC$  is double of the whole angle  $BAC$ .  
 Secondly, let the center of the circle be without the angle  $BAC$ .



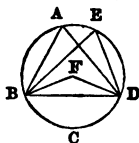
It may be demonstrated, as in the first case,  
 that the angle  $FEC$  is double of the angle  $FAC$ ,  
 and that  $FEB$ , a part of the first, is double of  $FAB$ , a part of the other;  
 therefore the remaining angle  $BEC$  is double of the remaining  
 angle  $BAC$ .

Therefore the angle at the center, &c. Q.E.D.

### PROPOSITION XXI. THEOREM.

*The angles in the same segment of a circle are equal to one another.*

Let  $ABCD$  be a circle,  
 and  $BAD, BED$  angles in the same segment  $BAED$ .  
 Then the angles  $BAD, BED$  shall be equal to one another.  
 First, let the segment  $BAED$  be greater than a semicircle.



Take  $F$ , the center of the circle  $ABCD$ , (III. 1.) and join  $BF, FD$ .

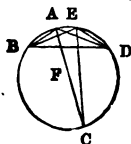
Because the angle  $BFD$  is at the center, and the angle  $BAD$  at the circumference, and that they have the same part of the circumference, viz. the arc  $BCD$  for their base;

therefore the angle  $BFD$  is double of the angle  $BAD$ : (III. 20.)

for the same reason the angle  $BFD$  is double of the angle  $BED$ :

therefore the angle  $BAD$  is equal to the angle  $BED$ . (ax. 7.)

Next, let the segment  $BAED$  be not greater than a semicircle.



Draw  $AF$  to the center, and produce it to  $C$ , and join  $CE$ .

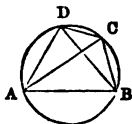
Because  $AC$  is a diameter of the circle,  
 therefore the segment  $BADC$  is greater than a semicircle;  
 and the angles in it  $BAC, BEC$  are equal, by the first case:

for the same reason, because  $CBED$  is greater than a semicircle,  
the angles  $CAD$ ,  $CED$ , are equal :  
therefore the whole angle  $BAD$  is equal to the whole angle  $BED$ . (ax. 2.)  
Wherefore the angles in the same segment, &c. Q. E. D.

## PROPOSITION XXII. THEOREM.

*The opposite angles of any quadrilateral figure inscribed in a circle, are together equal to two right angles.*

Let  $ABCD$  be a quadrilateral figure in the circle  $ABCD$ .  
Then any two of its opposite angles shall together be equal to two right angles.



Join  $AC$ ,  $BD$ .

And because the three angles of every triangle are equal to two right angles, (I. 32.)

the three angles of the triangle  $CAB$ , viz. the angles  $CAB$ ,  $ABC$ ,  $BCA$ , are equal to two right angles :

but the angle  $CAB$  is equal to the angle  $CDB$ , (III. 21.)

because they are in the same segment  $CADB$ ;

and the angle  $ACB$  is equal to the angle  $ADB$ ,

because they are in the same segment  $ADCB$  :

therefore the two angles  $CAB$ ,  $ACB$  are together equal to the whole angle  $ADC$ : (ax. 2.)

to each of these equals add the angle  $ABC$ ;

therefore the three angles  $ABC$ ,  $CAB$ ,  $BCA$  are equal to the two angles  $ABC$ ,  $ADC$ : (ax. 2.)

but  $ABC$ ,  $CAB$ ,  $BCA$ , are equal to two right angles;

therefore also the angles  $ABC$ ,  $ADC$  are equal to two right angles.

In the same manner, the angles  $BAD$ ,  $DCB$ , may be shewn to be equal to two right angles.

Therefore, the opposite angles, &c. Q. E. D.

## PROPOSITION XXIII. THEOREM.

*Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.*

If it be possible, upon the same straight line  $AB$ , and upon the same side of it, let there be two similar segments of circles,  $ACB$ ,  $ADB$ , not coinciding with one another.



Then, because the circumference  $ACB$  cuts the circumference  $ADB$  in the two points  $A, B$ , they cannot cut one another in any other point: (III. 10.)

therefore one of the segments must fall within the other:

let  $ACB$  fall within  $ADB$ :

draw the straight line  $BCD$ , and join  $CA, DA$ .

Because the segment  $ACB$  is similar to the segment  $ADB$ , (hyp.) and that similar segments of circles contain equal angles; (III. def. 11.)

therefore the angle  $ACB$  is equal to the angle  $ADB$ ,

the exterior angle to the interior, which is impossible. (I. 16.)

Therefore, there cannot be two similar segments of circles upon the same side of the same line, which do not coincide. Q.E.D.

#### PROPOSITION XXIV. THEOREM.

*Similar segments of circles upon equal straight lines, are equal to one another.*

Let  $AEB, CFD$  be similar segments of circles upon the equal straight lines  $AB, CD$ .

Then the segment  $AEB$  shall be equal to the segment  $CFD$ .



For if the segment  $AEB$  be applied to the segment  $CFD$ , so that the point  $A$  may be on  $C$ , and the straight line  $AB$  upon  $CD$ , then the point  $B$  shall coincide with the point  $D$ ,

because  $AB$  is equal to  $CD$ :

therefore, the straight line  $AB$  coinciding with  $CD$ , the segment  $AEB$  must coincide with the segment  $CFD$ , (III. 23.) and therefore is equal to it. (I. ax. 8.)

Wherefore similar segments, &c. Q.E.D.

#### PROPOSITION XXV. PROBLEM.

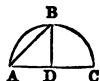
*A segment of a circle being given, to describe the circle of which it is the segment.*

Let  $ABC$  be the given segment of a circle.

It is required to describe the circle of which it is the segment.

Bisect  $AC$  in  $D$ , (I. 10.) and from the point  $D$  draw  $DB$  at right angles to  $AC$ , (I. 11.) and join  $AB$ .

First, let the angles  $ABD, BAD$  be equal to one another:



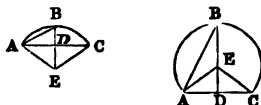
then the straight line  $DA$  is equal to  $DB$ , (I. 6.) and therefore, to  $DC$ ; and because the three straight lines  $DA, DB, DC$  are all equal, therefore  $D$  is the center of the circle. (III. 9.)

From the center  $D$ , at the distance of any of the three  $DA, DB, DC$ , describe a circle;

this shall pass through the other points;

and the circle of which  $ABC$  is a segment has been described:

and because the center  $D$  is in  $AC$ , the segment  $ABC$  is a semicircle.  
But if the angles  $ABD$ ,  $BAD$  are not equal to one another:



at the point  $A$ , in the straight line  $AB$ ,  
make the angle  $BAE$  equal to the angle  $ABD$ , (I. 23.)  
and produce  $BD$ , if necessary, to meet  $AE$  in  $E$ , and join  $EC$ .  
Because the angle  $ABE$  is equal to the angle  $BAE$ ,  
therefore the straight line  $EA$  is equal to  $EB$ : (I. 6.)  
and because  $AD$  is equal to  $DC$ , and  $DE$  common to the triangles  
 $ADE$ ,  $CDE$ ,  
the two sides  $AD$ ,  $DE$ , are equal to the two  $CD$ ,  $DE$ , each to each;  
and the angle  $ADE$  is equal to the angle  $CDE$ ;  
for each of them is a right angle; (constr.)  
therefore the base  $EA$  is equal to the base  $EC$ : (I. 4.)  
but  $EA$  was shewn to be equal to  $EB$ :  
wherefore also  $EB$  is equal to  $EC$ : (ax. 1.)  
and therefore the three straight lines  $EA$ ,  $EB$ ,  $EC$  are equal to one  
another:

wherefore  $E$  is the center of the circle. (III. 9.)  
From the center  $E$ , at the distance of any of the three  $EA$ ,  $EB$ ,  
 $EC$ , describe a circle;

this shall pass through the other points;  
and the circle of which  $ABC$  is a segment, is described.  
And it is evident, that if the angle  $ABD$  be greater than the angle  
 $BAD$ , the center  $E$  falls without the segment  $ABC$ , which therefore  
is less than a semicircle:

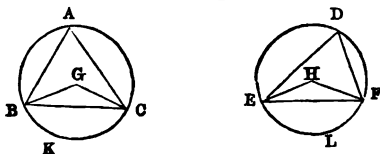
but if the angle  $ABD$  be less than  $BAD$ , the center  $E$  falls within  
the segment  $ABC$ , which is therefore greater than a semicircle.

Wherefore a segment of a circle being given, the circle is described  
of which it is a segment. Q. E. F.

# PROPOSITION XXVI. THEOREM.

*In equal circles, equal angles stand upon equal arcs, whether the angles be  
at the centers or circumferences.*

Let  $ABC$ ,  $DEF$  be equal circles,  
and let the angles  $BGC$ ,  $EHF$  at their centers,  
and  $BAC$ ,  $EDF$  at their circumferences be equal to each other.  
Then the arc  $BKC$  shall be equal to the arc  $ELF$ .



Join  $BC$ ,  $EF$ .

And because the circles  $ABC$ ,  $DEF$  are equal,  
the straight lines drawn from their centers are equal: (III. def. 1.)  
therefore the two sides  $BG$ ,  $GC$ , are equal to the two  $EH$ ,  $HF$ , each  
to each:

and the angle at  $G$  is equal to the angle at  $H$ ; (hyp.)

therefore the base  $BC$  is equal to the base  $EF$ . (I. 4.)

And because the angle at  $A$  is equal to the angle at  $D$ , (hyp.)  
the segment  $BAC$  is similar to the segment  $EDF$ : (III. def. 11.)

and they are upon equal straight lines  $BC$ ,  $EF$ :

but similar segments of circles upon equal straight lines, are equal to  
one another, (III. 24.)

therefore the segment  $BAC$  is equal to the segment  $EDF$ :

but the whole circle  $ABC$  is equal to the whole  $DEF$ ; (hyp.)

therefore the remaining segment  $BKC$  is equal to the remaining seg-  
ment  $ELF$ , (I. ax. 3.)

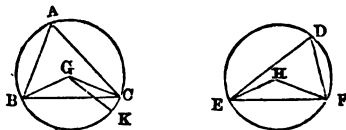
and the arc  $BKC$  to the arc  $ELF$ .

Wherefore, in equal circles, &c. Q. E. D.

### PROPOSITION XXVII. THEOREM.

*In equal circles, the angles which stand upon equal arcs, are equal to one another, whether they be at the centers or circumferences.*

Let  $ABC$ ,  $DEF$  be equal circles,  
and let the angles  $BGC$ ,  $EHF$  at their centers,  
and the angles  $BAC$ ,  $EDF$  at their circumferences,  
stand upon the equal arcs  $BC$ ,  $EF$ .  
Then the angle  $BGC$  shall be equal to the angle  $EHF$ ,  
and the angle  $BAC$  to the angle  $EDF$ .



If the angle  $BGC$  be equal to the angle  $EHF$ ,  
it is manifest that the angle  $BAC$  is also equal to  $EDF$ . (III. 20. and  
I. ax. 7.)

But, if not, one of them must be greater than the other:

if possible, let the angle  $BGC$  be greater than  $EHF$ ,

and at the point  $G$ , in the straight line  $BG$ ,

make the angle  $BGK$  equal to the angle  $EHF$ . (I. 23.)

Then because the angle  $BGK$  is equal to the angle  $EHF$ ,  
and that equal angles stand upon equal arcs, when they are at the  
centers; (III. 26.)

therefore the arc  $BK$  is equal to the arc  $EF$ :

but the arc  $EF$  is equal to the arc  $BC$ ; (hyp.)

therefore also the arc  $BK$  is equal to the arc  $BC$ ,

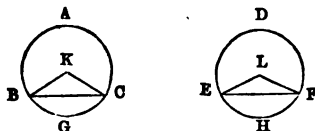
the less equal to the greater, which is impossible: (I. ax. 1.)

therefore the angle  $BGC$  is not unequal to the angle  $EHF$ ;  
 that is, it is equal to it:  
 but the angle at  $A$  is half of the angle  $BGC$ , (III. 20.)  
 and the angle at  $D$ , half of the angle  $EHF$ ;  
 therefore the angle at  $A$  is equal to the angle at  $D$ . (I. ax. 7.)  
 Wherefore, in equal circles, &c. Q. E. D.

PROPOSITION XXVIII. THEOREM.

*In equal circles, equal straight lines cut off equal arcs, the greater equal to the greater, and the less to the less.*

Let  $ABC$ ,  $DEF$  be equal circles,  
 and  $BC$ ,  $EF$  equal straight lines in them, which cut off the two greater  
 arcs  $BAC$ ,  $EDF$ , and the two less  $BGC$ ,  $EHF$ .  
 Then the greater arc  $BAC$  shall be equal to the greater  $EDF$ ,  
 and the less arc  $BGC$  to the less  $EHF$ .



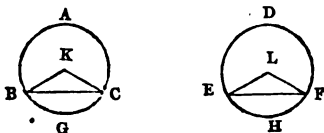
Take  $K$ ,  $L$ , the centers of the circles, (III. 1.) and join  $BK$ ,  $KC$ ,  $EL$ ,  $LF$ .

Because the circles  $ABC$ ,  $DEF$  are equal,  
 the straight lines from their centers are equal: (III. def. 1.)  
 therefore  $BK$ ,  $KC$  are equal to  $EL$ ,  $LF$ , each to each:  
 and the base  $BC$  is equal to the base  $EF$ , in the triangles  $BCK$ ,  $EFL$ ;  
 therefore the angle  $BKC$  is equal to the angle  $ELF$ : (I. 8.)  
 but equal angles stand upon equal arcs, when they are at the  
 centers: (III. 26.)  
 therefore the arc  $BGC$  is equal to the arc  $EHF$ :  
 but the whole circumference  $ABC$  is equal to the whole  $EDF$ ; (hyp.)  
 therefore the remaining part of the circumference,  
 viz. the arc  $BAC$ , is equal to the remaining part  $EDF$ . (I. ax. 3.)  
 Therefore, in equal circles, &c. Q. E. D.

PROPOSITION XXIX. THEOREM.

*In equal circles, equal arcs are subtended by equal straight lines.*

Let  $ABC$ ,  $DEF$  be equal circles,  
 and let the arcs  $BGC$ ,  $EHF$  also be equal,  
 and joined by the straight lines  $BC$ ,  $EF$ .  
 Then the straight line  $BC$  shall be equal to the straight line  $EF$ .





Take  $K, L$ , (III. 1.) the centers of the circles, and join  $BK, KC, EL, LF$ .

Because the arc  $BGC$  is equal to the arc  $EHF$ ,  
therefore the angle  $BKC$  is equal to the angle  $ELF$ : (III. 27.)

and because the circles  $ABC, DEF$  are equal,  
the straight lines from their centers are equal; (III. def. 1.)

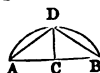
therefore  $BK, KC$  are equal to  $EL, LF$ , each to each;  
and they contain equal angles in the triangles  $BCK, EFL$ ;  
therefore the base  $BC$  is equal to the base  $EF$ . (I. 4.)

Therefore, in equal circles, &c. Q. E. D.

### PROPOSITION XXX. PROBLEM.

To bisect a given arc, that is, to divide it into two equal parts.

Let  $ADB$  be the given arc:  
it is required to bisect it.



Join  $AB$ , and bisect it in  $C$ ; (I. 10.)

from the point  $C$  draw  $CD$  at right angles to  $AB$ . (I. 11.)

Then the arc  $ADB$  shall be bisected in the point  $D$ .

Join  $AD, DB$ .

And because  $AC$  is equal to  $CB$ ,  
and  $CD$  common to the triangles  $ACD, BCD$ ,  
the two sides  $AC, CD$  are equal to the two  $BC, CD$ , each to each;  
and the angle  $ACD$  is equal to the angle  $BCD$ ,  
because each of them is a right angle:

therefore the base  $AD$  is equal to the base  $BD$ . (I. 4.)

But equal straight lines cut off equal arcs, (III. 28.)  
the greater arc equal to the greater, and the less arc to the less;  
and the arcs  $AD, DB$  are each of them less than a semicircle;  
because  $DC$ , if produced, passes through the center: (III. 1. Cor.)  
therefore the arc  $AD$  is equal to the arc  $DB$ .

Therefore the given arc  $ADB$  is bisected in  $D$ . Q. E. F.

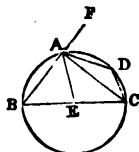
### PROPOSITION XXXI. THEOREM.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let  $ABCD$  be a circle, of which the diameter is  $BC$ , and center  $E$ ,  
and let  $CA$  be drawn, dividing the circle into the segments  $ABC, ADC$ .

Join  $BA, AD, DC$ .

Then the angle in the semicircle  $BAC$  shall be a right angle;  
and the angle in the segment  $ABC$ , which is greater than a semicircle,  
shall be less than a right angle;  
and the angle in the segment  $ADC$ , which is less than a semicircle,  
shall be greater than a right angle.



Join  $AE$ , and produce  $BA$  to  $F$ .

First, because  $EB$  is equal to  $EA$ , (I. def. 15.)

the angle  $EAB$  is equal to  $EBA$ ; (I. 5.)

also, because  $EA$  is equal to  $EC$ ,

the angle  $ECA$  is equal to  $EAC$ ;  
wherefore the whole angle  $BAC$  is equal to the two angles  $EBA$ ,  
 $ECA$ ; (1. ax. 2.)

but  $\angle FAC$ , the exterior angle of the triangle  $ABC$ , is equal to the two angles  $EBA$ ,  $ECA$ ; (I. 32.)

therefore the angle  $BAC$  is equal to the angle  $FAC$ ; (ax. 1.)

and therefore each of them is a right angle: (I. def. 10.)

wherefore the angle  $BAC$  in a semicircle is a right angle

Secondly, because the two angles  $ABC, BAC$  of the triangle  $ABC$  are together less than two right angles, (I. 17.)

and that  $BAC$  has been proved to be a right angle;

therefore  $ABC$  must be less than a right angle:

and therefore the angle in a segment  $ABC$  greater than a semicircle, is less than a right angle.

And lastly, because  $ABCD$  is a quadrilateral figure in a circle,

any two of its opposite angles are equal to two right angles: (III. 22.)

therefore the angles  $ABC, ADC$ , are equal to two right angles:

and  $ABC$  has been proved to be less than a right angle ;

wherefore the other  $ADC$  is greater than a right angle.

Therefore, in a circle the angle in a semicircle is a right angle; &c. Q.E.D.

COR. From this it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle: because the angle adjacent to it is equal to the same two; (I. 32.) and when the adjacent angles are equal, they are right angles. (I. def. 10.)

**PROPOSITION XXXII. THEOREM.**

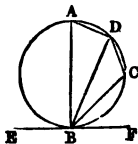
*If a straight line touch a circle, and from the point of contact a straight line be drawn meeting the circle; the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.*

Let the straight line  $EF$  touch the circle  $ABCD$  in  $B$ , and from the point  $B$  let the straight line  $BD$  be drawn, meeting the circumference in  $D$ , and dividing it into the segments  $DCB$ ,  $DAB$ , of which  $DCB$  is less than, and  $DAB$  greater than a semicircle.

Then the angles which  $BD$  makes with the touching line  $EF$ , shall be equal to the angles in the alternate segments of the circle;

that is, the angle  $\angle DBF$  shall be equal to the angle which is in the segment  $DAB$ ,

and the angle  $DBE$  shall be equal to the angle in the alternate segment  $DCB$ .



From the point  $B$  draw  $BA$  at right angles to  $EF$ , (I. 11.) meeting the circumference in  $A$ ;

take any point  $C$  in the arc  $DB$ , and join  $AD$ ,  $DC$ ,  $CB$ .

Because the straight line  $EF$  touches the circle  $ABCD$  in the point  $B$ ,

and  $BA$  is drawn at right angles to the touching line from the point of contact  $B$ ,

the center of the circle is in  $BA$ : (III. 19.)

therefore the angle  $ADB$  in a semicircle is a right angle: (III. 31.)

and consequently the other two angles  $BAD$ ,  $ABD$ , are equal to a right angle; (I. 32.)

but  $ABF$  is likewise a right angle; (constr.)

therefore the angle  $ABF$  is equal to the angles  $BAD$ ,  $ABD$ : (I. ax. 1.)

take from these equals the common angle  $ABD$ :

therefore the remaining angle  $DBF$  is equal to the angle  $BAD$ , (I. ax. 3.)

which is in  $BDA$ , the alternate segment of the circle.

And because  $ABCD$  is a quadrilateral figure in a circle,

the opposite angles  $BAD$ ,  $BCD$  are equal to two right angles: (III. 22.)

but the angles  $DBF$ ,  $DBE$  are likewise equal to two right angles; (I. 13.)

therefore the angles  $DBF$ ,  $DBE$  are equal to the angles  $BAD$ ,  $BCD$ , (I. ax. 1.)

and  $DBF$  has been proved equal to  $BAD$ ;

therefore the remaining angle  $DBE$  is equal to the angle  $BCD$  in  $BDC$ , the alternate segment of the circle. (I. ax. 2.)

Wherefore, if a straight line, &c. Q. E. D.

### PROPOSITION XXXIII. PROBLEM.

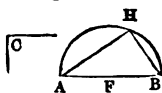
Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

Let  $AB$  be the given straight line,

and the angle  $C$  the given rectilineal angle.

It is required to describe upon the given straight line  $AB$ , a segment of a circle, which shall contain an angle equal to the angle  $C$ .

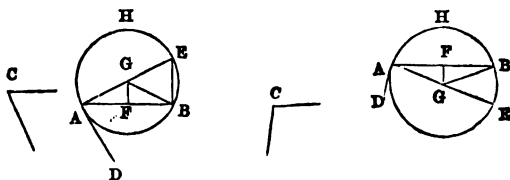
First, let the angle  $C$  be a right angle.



Bisect  $AB$  in  $F$ , (I. 10.)  
and from the center  $F$ , at the distance  $FB$ , describe the semicircle  $AHB$ ,  
and draw  $AH$ ,  $BH$  to any point  $H$  in the circumference.

Therefore the angle  $AHB$  in a semicircle is equal to the right angle  $C$ . (III. 31.)

But if the angle  $C$  be not a right angle:



at the point  $A$ , in the straight line  $AB$ ,  
make the angle  $BAD$  equal to the angle  $C$ , (I. 23.)  
and from the point  $A$  draw  $AE$  at right angles to  $AD$ ; (I. 11.)  
bisect  $AB$  in  $F$ , (I. 10.)  
and from  $F$  draw  $FG$  at right angles to  $AB$ , (I. 11.) and join  $GB$ .  
Because  $AF$  is equal to  $FB$ , and  $FG$  common to the triangles  
 $AFG$ ,  $BFG$ ,  
the two sides  $AF$ ,  $FG$  are equal to the two  $BF$ ,  $FG$ , each to each,  
and the angle  $AFG$  is equal to the angle  $BFG$ ; (I. def. 10.)  
therefore the base  $AG$  is equal to the base  $GB$ ; (I. 4.)  
and the circle described from the center  $G$ , at the distance  $GA$ ,  
shall pass through the point  $B$ :  
let this be the circle  $AHB$ .

The segment  $AHB$  shall contain an angle equal to the given rectilineal angle  $C$ .

Because from the point  $A$  the extremity of the diameter  $AE$ ,  
 $AD$  is drawn at right angles to  $AE$ ,

therefore  $AD$  touches the circle: (III. 16. Cor.)

and because  $AB$ , drawn from the point of contact  $A$ , cuts the circle,  
the angle  $DAB$  is equal to the angle in the alternate segment  
 $AHB$ : (III. 32.)

but the angle  $DAB$  is equal to the angle  $C$ ; (constr.)

therefore the angle  $C$  is equal to the angle in the segment  $AHB$ .

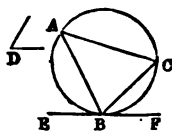
Wherefore, upon the given straight line  $AB$ , the segment  $AHB$   
of a circle is described, which contains an angle equal to the given  
angle  $C$ . Q.E.F.

# PROPOSITION XXXIV. PROBLEM.

From a given circle to cut off a segment, which shall contain an angle  
equal to a given rectilineal angle.

Let  $ABC$  be the given circle, and  $D$  the given rectilineal angle.

It is required to cut off from the circle  $ABC$  a segment that shall  
contain an angle equal to the given angle  $D$ .



Draw the straight line  $EF$  touching the circle  $ABC$  in any point  $B$ , (III. 17.)

and at the point  $B$ , in the straight line  $BF$ ,  
make the angle  $FBC$  equal to the angle  $D$ . (I. 23.)

Then the segment  $BAC$  shall contain an angle equal to the given angle  $D$ .

Because the straight line  $EF$  touches the circle  $ABC$ ,

and  $BC$  is drawn from the point of contact  $B$ ,  
therefore the angle  $FBC$  is equal to the angle in the alternate  
segment  $BAC$  of the circle: (III. 32.)

but the angle  $FBC$  is equal to the angle  $D$ ; (constr.)

therefore the angle in the segment  $BAC$  is equal to the angle  
 $D$ . (I. ax. 1.)

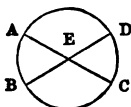
Wherefore from the given circle  $ABC$ , the segment  $BAC$  is cut  
off, containing an angle equal to the given angle  $D$ . Q. E. F.

### PROPOSITION XXXV. THEOREM.

*If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them, is equal to the rectangle contained by the segments of the other.*

Let the two straight lines  $AC, BD$ , cut one another in the point  
 $E$ , within the circle  $ABCD$ .

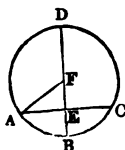
Then the rectangle contained by  $AE, EC$  shall be equal to the  
rectangle contained by  $BE, ED$ .



First, if  $AC, BD$  pass each of them through the center, so that  $E$   
is the center;

it is evident that since  $AE, EC, BE, ED$ , being all equal, (I. def. 15.)  
therefore the rectangle  $AE, EC$  is equal to the rectangle  $BE, ED$ .

Secondly, let one of them  $BD$  pass through the center, and cut the  
other  $AC$ , which does not pass through the center, at right angles, in  
the point  $E$ .



Then, if  $BD$  be bisected in  $F$ ,  
 $F$  is the center of the circle  $ABCD$ .

Join  $AF$ .

Because  $BD$  which passes through the center, cuts the straight line  $AC$ , which does not pass through the center, at right angles in  $E$ , therefore  $AE$  is equal to  $EC$ : (III. 3.)

and because the straight line  $BD$  is cut into two equal parts in the point  $F$ , and into two unequal parts in the point  $E$ , therefore the rectangle  $BE, ED$ , together with the square on  $EF$ , is equal to the square on  $FB$ : (II. 5.)

that is, to the square on  $FA$ :

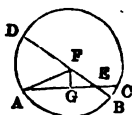
but the squares on  $AE, EF$ , are equal to the square on  $FA$ : (I. 47.) therefore the rectangle  $BE, ED$ , together with the square on  $EF$ , is equal to the squares on  $AE, EF$ : (I. ax. 1.)

take away the common square on  $EF$ ,

and the remaining rectangle  $BE, ED$  is equal to the remaining square on  $AE$ : (I. ax. 3.)

that is, to the rectangle  $AE, EC$ .

Thirdly, let  $BD$ , which passes through the center, cut the other  $AC$ , which does not pass through the center, in  $E$ , but not at right angles.



Then, as before, if  $BD$  be bisected in  $F$ ,

$F$  is the center of the circle.

Join  $AF$ , and from  $F$  draw  $FG$  perpendicular to  $AC$ : (I. 12.)

therefore  $AG$  is equal to  $GC$ : (III. 3.)

wherefore the rectangle  $AE, EC$ , together with the square on  $EG$ , is equal to the square on  $AG$ : (II. 5.)

to each of these equals add the square on  $GF$ ;

therefore the rectangle  $AE, EC$ , together with the squares on  $EG, GF$ , is equal to the squares on  $AG, GF$ : (I. ax. 2.)

but the squares on  $EG, GF$ , are equal to the square on  $EF$ : (I. 47.)

and the squares on  $AG, GF$  are equal to the square on  $AF$ :

therefore the rectangle  $AE, EC$ , together with the square on  $EF$ , is equal to the square on  $AF$ ;

that is, to the square on  $FB$ :

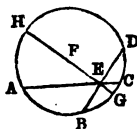
but the square on  $FB$  is equal to the rectangle  $BE, ED$ , together with the square on  $EF$ : (II. 5.)

therefore the rectangle  $AE, EC$ , together with the square on  $EF$ , is equal to the rectangle  $BE, ED$ , together with the square on  $EF$ : (I. ax. 1.)

take away the common square on  $EF$ ,

and the remaining rectangle  $AE, EC$ , is therefore equal to the remaining rectangle  $BE, ED$ . (ax. 3.)

Lastly, let neither of the straight lines  $AC, BD$  pass through the center.



Take the center  $F$ , (III. 1.)

and through  $E$  the intersection of the straight lines  $AC$ ,  $DB$ ,  
draw the diameter  $GEFH$ .

And because the rectangle  $AE$ ,  $EC$  is equal, as has been shewn,  
to the rectangle  $GE$ ,  $EH$ ;

and for the same reason, the rectangle  $BE$ ,  $ED$  is equal to the  
same rectangle  $GE$ ,  $EH$ ;

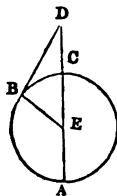
therefore the rectangle  $AE$ ,  $EC$  is equal to the rectangle  $BE$ ,  $ED$ .  
(I. AX. 1.)

Wherefore, if two straight lines, &c. Q.E.D.

### PROPOSITION XXXVI. THEOREM.

*If from any point without a circle two straight lines be drawn, one of  
which cuts the circle, and the other touches it; the rectangle contained by  
the whole line which cuts the circle, and the part of it without the circle,  
shall be equal to the square on the line which touches it.*

Let  $D$  be any point without the circle  $ABC$ ,  
and let  $DCA$ ,  $DB$  be two straight lines drawn from it,  
of which  $DCA$  cuts the circle, and  $DB$  touches the same.  
Then the rectangle  $AD$ ,  $DC$  shall be equal to the square on  $DB$ .  
Either  $DCA$  passes through the center, or it does not:  
first, let it pass through the center  $E$ .



Join  $EB$ ,

therefore the angle  $EBD$  is a right angle. (III. 18.)

And because the straight line  $AC$  is bisected in  $E$ , and produced  
to the point  $D$ ,  
therefore the rectangle  $AD$ ,  $DC$ , together with the square on  $EC$ , is  
equal to the square on  $ED$ : (II. 6.)

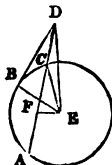
but  $CE$  is equal to  $EB$ ;

therefore the rectangle  $AD$ ,  $DC$ , together with the square on  $EB$ ,  
is equal to the square on  $ED$ :  
but the square on  $ED$  is equal to the squares on  $EB$ ,  $BD$ , (I. 47.)

because  $EBD$  is a right angle:

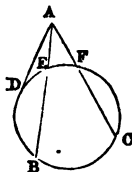
therefore the rectangle  $AD$ ,  $DC$ , together with the square on  $EB$ ,  
is equal to the squares on  $EB$ ,  $BD$ : (AX. 1.)

take away the common square on  $EB$ ;  
therefore the remaining rectangle  $AD, DC$  is equal to the square  
on the tangent  $DB$ . (ax. 3.)  
Next, if  $DCA$  does not pass through the center of the circle  $ABC$ .



Take  $E$  the center of the circle, (III. 1.)  
draw  $EF$  perpendicular to  $AC$ , (I. 12.) and join  $EB, EC, ED$ .  
Because the straight line  $EF$ , which passes through the center,  
cuts the straight line  $AC$ , which does not pass through the center, at  
right angles; it also bisects  $AC$ , (III. 3.)  
therefore  $AF$  is equal to  $FC$ ;  
and because the straight line  $AC$  is bisected in  $F$ , and produced to  $D$ ,  
the rectangle  $AD, DC$ , together with the square on  $FC$ ,  
is equal to the square on  $FD$ : (II. 6.)  
to each of these equals add the square on  $FE$ ;  
therefore the rectangle  $AD, DC$ , together with the squares on  $CF, FE$ ,  
is equal to the squares on  $DF, FE$ : (I. ax. 2.)  
but the square on  $ED$  is equal to the squares on  $DF, FE$ , (I. 47.)  
because  $EFD$  is a right angle;  
and for the same reason,  
the square on  $EC$  is equal to the squares on  $CF, FE$ ;  
therefore the rectangle  $AD, DC$ , together with the square on  $EC$ ,  
is equal to the square on  $ED$ : (ax. 1.)  
but  $CE$  is equal to  $EB$ ;  
therefore the rectangle  $AD, DC$ , together with the square on  $EB$ ,  
is equal to the square on  $ED$ :  
but the squares on  $EB, BD$ , are equal to the square on  $ED$ , (I. 47.)  
because  $EBD$  is a right angle:  
therefore the rectangle  $AD, DC$ , together with the square on  $EB$ ,  
is equal to the squares on  $EB, BD$ ;  
take away the common square on  $EB$ ;  
and the remaining rectangle  $AD, DC$  is equal to the square  
on  $DB$ . (I. ax. 3.)

Wherefore, if from any point, &c. Q. E. D.  
COR. If from any point without a circle, there be drawn two straight





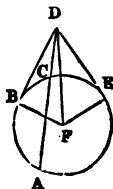
lines cutting it, as  $AB$ ,  $AC$ , the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle  $BA$ ,  $AE$ , to the rectangle  $CA$ ,  $AF$ : for each of them is equal to the square on the straight line  $AD$ , which touches the circle.

### PROPOSITION XXXVII. THEOREM.

*If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets it, the line which meets, shall touch the circle.*

Let any point  $D$  be taken without the circle  $ABC$ , and from it let two straight lines  $DCA$  and  $DB$  be drawn, of which  $DCA$  cuts the circle in the points  $C$ ,  $A$ , and  $DB$  meets it in the point  $B$ .

If the rectangle  $AD$ ,  $DC$  be equal to the square on  $DB$ ;  
then  $DB$  shall touch the circle.



Draw the straight line  $DE$ , touching the circle  $ABC$ , in the point  $E$ ; (III. 17.)

find  $F$ , the center of the circle, (III. 1.)  
and join  $FE$ ,  $FB$ ,  $FD$ .

Then  $FED$  is a right angle: (III. 18.)

and because  $DE$  touches the circle  $ABC$ , and  $DCA$  cuts it, therefore the rectangle  $AD$ ,  $DC$  is equal to the square on  $DE$ : (III. 36.)

but the rectangle  $AD$ ,  $DC$ , is, by hypothesis,  
equal to the square on  $DB$ :

therefore the square on  $DE$  is equal to the square on  $DB$ ; (I. ax. 1.)

and the straight line  $DE$  equal to the straight line  $DB$ :

and  $FE$  is equal to  $FB$ ; (I. def. 15.)

wherefore  $DE$ ,  $EF$  are equal to  $DB$ ,  $BF$ , each to each;

and the base  $FD$  is common to the two triangles  $DEF$ ,  $DBF$ ;

therefore the angle  $DEF$  is equal to the angle  $DBF$ : (I. 8.)

but  $DEF$  was shewn to be a right angle;

therefore also  $DBF$  is a right angle: (I. ax. 1.)

and  $BF$ , if produced, is a diameter;

and the straight line which is drawn at right angles to a diameter,  
from the extremity of it, touches the circle; (III. 16. Cor.)

therefore  $DB$  touches the circle  $ABC$ .

Wherefore, if from a point, &c. Q. E. D

## NOTES TO BOOK III.

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In the Third Book of the Elements are demonstrated the most elementary properties of the circle, assuming all the properties of figures demonstrated in the First and Second Books.

It may be worthy of remark, that the word *circle* will be found sometimes taken to mean *the surface* included within the circumference, and sometimes *the circumference itself*. Euclid has employed the word ( $\pi\epsilon\rho\iota\phi\acute{\epsilon}\rho\epsilon\iota\alpha$ ) *periphery*, both for the whole, and for a part of the circumference of a circle. If the word *circumference* were restricted to mean *the whole* circumference, and the word *arc* to mean a *part of it*, ambiguity might be avoided when speaking of the circumference of a circle, where only a part of it is the subject under consideration. A circle is said to be given in position, when the position of its center is known, and in magnitude, when its radius is known.

Def. 1. And it may be added, or of which the circumferences are equal. And conversely: if two circles be equal, their diameters and radii are equal; as also their circumferences.

Def. 1. states the criterion of equal circles. Simson calls it a theorem; and Euclid seems to have considered it as one of those theorems, or axioms, which might be admitted as a basis for reasoning on the equality of circles.

Def. 11. There seems to be tacitly assumed in this definition, that a straight line, when it meets a circle and does not touch it, must necessarily, when produced, cut the circle.

A straight line which touches a circle, is called a *tangent* to the circle; and a straight line which cuts a circle is called a *secant*.

Def. 14. The distance of a straight line from the center of a circle is the distance of a point from a straight line, which has been already explained in note to Prop. 11. page 53.

Def. 16. 1. An *arc* of a circle is any portion of the circumference; and a *chord* is the straight line joining the extremities of an arc. Every chord except a diameter divides a circle into two unequal segments, one greater than, and the other less than a semicircle. And in the same manner, two radii drawn from the center to the circumference, divide the circle into two unequal sectors, which become equal when the two radii are in the same straight line. As Euclid, however, does not notice re-entering angles, a sector of the circle seems necessarily restricted to the figure which is less than a semicircle. A quadrant is a sector whose radii are perpendicular to one another, and which contains a fourth part of the circle.

Def. 17. No use is made of this definition in the Elements.

Def. 21. The definition of similar segments of circles as employed in the Third Book is restricted to such segments as are also equal. Props. 33.11. and 33.14. are the only two instances, in which reference is made to similar segments of circles.

Prop. 1. "Lines drawn in a circle," always mean in Euclid, such lines only as are terminated at their extremities by the circumference.

If the point  $G$  be in the diameter  $CE$ , but not coinciding with the point  $F$ , the demonstration given in the text does not hold good. At the same time, it is obvious that  $G$  cannot be the centre of the circle, because  $GC$  is not equal to  $GE$ .

*Indirect* demonstrations are more frequently employed in the Third Book than in the First Book of the Elements. Of the demonstrations of the forty-eight propositions of the First Book, nine are indirect: but of the thirty-seven of the Third Book, no less than fifteen are indirect demonstrations. The *indirect* is, in general, less readily appreciated by the learner, than the *direct* form of demonstration. The indirect form, however, is equally satisfactory, as it excludes every assumed hypothesis as false, except that which is made in the enunciation of the proposition. It may be here remarked that Euclid employs three methods of demonstrating converse propositions. First, by indirect demonstrations as in *Euc. i. 6: iii. 1, &c.* Secondly, by shewing that neither side of a possible alternative can be true, and thence inferring the truth of the proposition, as in *Euc. i. 19, 25.* Thirdly, by means of a construction, thereby avoiding the indirect mode of demonstration, as in *Euc. i. 47. iii. 37.*

**Prop. II.** In this proposition, the circumference of a circle is proved to be essentially different from a straight line, by shewing that every straight line joining any two points in the arc falls entirely within the circle, and can neither coincide with any part of the circumference, nor meet it except in the two assumed points. It excludes the idea of the circumference of a circle being flexible, or capable under any circumstances, of admitting the possibility of the line falling outside the circle.

If the line could fall partly within and partly without the circle, the circumference of the circle would intersect the line at some point between its extremities, and any part *without* the circle has been shewn to be impossible, and the part *within* the circle is in accordance with the enunciation of the Proposition. If the line could fall upon the circumference and coincide with it, it would follow that a straight line coincides with a curved line.

From this proposition follows the corollary, that "a straight line cannot cut the circumference of a circle in more points than two."

Commandine's direct demonstration of **Prop. II.** depends on the following axiom, "If a point be taken nearer to the center of a circle than the circumference, that point falls within the circle."

Take any point *E* in *AB*, and join *DA, DE, DB.* (fig. *Euc. iii. 2.*)

Then because *DA* is equal to *DB* in the triangle *DAB*;

therefore the angle *DAB* is equal to the angle *DBA*; (*i. 5.*)

but since the side *AE* of the triangle *DAE* is produced to *E*, therefore the exterior angle *DEB* is greater than the interior and opposite angle *DAE*; (*i. 16.*)

but the angle *DAE* is equal to the angle *DBE*,

therefore the angle *DEB* is greater than the angle *DBE*.

And in every triangle, the greater side is subtended by the greater angle;

therefore the side *DB* is greater than the side *DE*;

but *DB* from the center meets the circumference of the circle,

therefore *DE* does not meet it.

Wherefore the point *E* falls within the circle:

and *E* is any point in the straight line *AB*:

therefore the straight line *AB* falls within the circle.

**Prop. VII.** and **Prop. VIII.** exhibit the same property; in the former, the point is taken in the diameter, and in the latter, in the diameter produced.

**Prop. VIII.** An arc of a circle is said to be *convex* or *concave* with respect to a point, according as the straight lines drawn from the point

meet the *outside* or *inside* of the circular arc: and the two points found in the circumference of a circle by two straight lines drawn from a given point to touch the circle, divide the circumference into two portions, one of which is *convex* and the other *concave*, with respect to the given point.

Prop. ix. This appears to follow as a Corollary from Euc. III. 7.

Prop. xi. and Prop. xii. In the enunciation it is not asserted that the contact of two circles is confined to a single point. The meaning appears to be, that supposing two circles to touch each other in any point, the straight line which joins their centers being produced, shall pass through that point in which the circles touch each other. In Prop. xiii. it is proved that a circle cannot touch another in more points than one, by assuming two points of contact, and proving that this is impossible.

Prop. xiii. The following is Euclid's demonstration of the case, in which one circle touches another on the inside.

If possible, let the circle  $EBF$  touch the circle  $ABC$  on the inside, in more points than in one point, namely in the points  $B, D$ . (fig. Euc. III. 13.) Let  $P$  be the center of the circle  $ABC$ , and  $Q$  the center of  $EBF$ . Join  $P, Q$ ; then  $PQ$  produced shall pass through the points of contact  $B, D$ . For since  $P$  is the center of the circle  $ABC$ ,  $PB$  is equal to  $PD$ , but  $PB$  is greater than  $QD$ , much more then is  $QB$  greater than  $QD$ . Again, since the point  $Q$  is the center of the circle  $EBF$ ,  $QB$  is equal to  $QD$ ; but  $QB$  has been shewn to be greater than  $QD$ , which is impossible. One circle therefore cannot touch another on the inside in more points than in one point.

Prop. xvi. may be demonstrated *directly* by assuming the following axiom; "If a point be taken further from the center of a circle than the circumference, that point falls without the circle."

If one circle touch another, either *internally* or *externally*, the two circles can have, at the point of contact, only one common tangent.

Prop. xvii. When the given point is without the circumference of the given circle, it is obvious that two equal tangents may be drawn from the given point to touch the circle, as may be seen from the diagram to Prop. viii.

The best practical method of drawing a tangent to a circle from a given point without the circumference, is the following: join the given point and the center of the circle, upon this line describe a semicircle cutting the given circle, then the line drawn from the given point to the intersection will be the tangent required.

Circles are called *concentric circles* when they have the same center.

Prop. xviii. appears to be nothing more than the converse to Prop. xvi., because a tangent to any point of a circumference of a circle is a straight line at right angles at the extremity of the diameter which meets the circumference in that point.

Prop. xx. This proposition is proved by Euclid only in the case in which the angle at the circumference is less than a right angle, and the demonstration is free from objection. If, however, the angle at the circumference be a right angle, the angle at the center disappears, by the two straight lines from the center to the extremities of the arc becoming one straight line. And, if the angle at the circumference be an obtuse angle, the angle formed by the two lines from the center, does not stand on the same arc, but upon the arc which the assumed arc wants of the whole circumference.

If Euclid's definition of an angle be strictly observed, Prop. xx. is geometrically true, only when the angle at the center is less than two

right angles. If, however, the defect of an angle from four right angles may be regarded as an angle, the proposition is universally true, as may be proved by drawing a line from the angle in the circumference through the center, and thus forming two angles at the center, in Euclid's strict sense of the term.

In the first case, it is assumed that, if there be four magnitudes, such that the first is double of the second, and the third double of the fourth, then the first and third together shall be double of the second and fourth together: also in the second case, that if one magnitude be double of another, and a part taken from the first be double of a part taken from the second, the remainder of the first shall be double the remainder of the second, which is, in fact, a particular case of Prop. v. Book v.

Prop. xxi. Hence, the locus of the vertices of all triangles upon the same base, and which have the same vertical angle, is a circular arc.

Prop. xxii. The converse of this Proposition, namely: If the opposite angles of a quadrilateral figure be equal to two right angles, a circle can be described about it, is not proved by Euclid.

It is obvious from the demonstration of this proposition, that if any side of the inscribed figure be produced, the exterior angle is equal to the opposite angle of the figure.

Prop. xxiii. It is obvious from this proposition that of two circular segments upon the same base, the larger is that which contains the smaller angle.

Prop. xxv. The three cases of this proposition may be reduced to one, by drawing any two contiguous chords to the given arc, bisecting them, and from the points of bisection drawing perpendiculars. The point in which they meet will be the center of the circle. This problem is equivalent to that of finding a point equally distant from three given points.

Props. xxvi—xxix. The properties predicated in these four propositions with respect to *equal circles*, are also true when predicated of the *same circle*.

Prop. xxxi. suggests a method of drawing a line at right angles to another when the given point is at the extremity of the given line. And that if the diameter of a circle be one of the equal sides of an isosceles triangle, the base is bisected by the circumference.

Prop. xxxv. The most general case of this Proposition might have been first demonstrated, and the other more simple cases deduced from it. But this is not Euclid's method. He always commences with the more simple case and proceeds to the more difficult afterwards. The following process is the reverse of Euclid's method.

Assuming the construction in the last fig. to *Euc. iii. 35*. Join *FA*, *FD*, and draw *FK* perpendicular to *AC*, and *FL* perpendicular to *BD*. Then (*Euc. ii. 5*.) the rectangle *AE*, *EC* with square on *EK* is equal to the square on *AK*: add to these equals the square on *FK*: therefore the rectangle *AE*, *EC*, with the squares on *EK*, *FK*, is equal to the squares on *AK*, *FK*. But the squares on *EK*, *FK* are equal to the square on *EF*, and the squares on *AK*, *FK* are equal to the square on *AF*. Hence the rectangle *AE*, *EC*, with the square on *EF* is equal to the square on *AF*.

In a similar way may be shewn, that the rectangle *BE*, *ED* with the square on *EF* is equal to the square on *FD*. And the square on *FD* is equal to the square on *AD*. Wherefore the rectangle *AE*, *EC* with the square on *EF* is equal to the rectangle *BE*, *ED* with the square on *EF*. Take from these equals the square on *EF*, and the rectangle *AE*, *EC* is equal to the rectangle *BE*, *ED*.

The other more simple cases may easily be deduced from this general case.

The converse is not proved by Euclid; namely,—If two straight lines intersect one another, so that the rectangle contained by the parts of one is equal to the rectangle contained by the parts of the other; then a circle may be described passing through the extremities of the two lines. Or, in other words:—If the diagonals of a quadrilateral figure intersect one another, so that the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other; then a circle may be described about the quadrilateral.

Prop. xxxvi. The converse of the corollary to this proposition may be thus stated:—If there be two straight lines, such that, when produced to meet, the rectangle contained by one of the lines produced, and the part produced, be equal to the rectangle contained by the other line produced and the part produced; then a circle can be described passing through the extremities of the two straight lines. Or, If two opposite sides of a quadrilateral figure be produced to meet, and the rectangle contained by one of the sides produced and the part produced, be equal to the rectangle contained by the other side produced and the part produced; then a circle may be described about the quadrilateral figure.

Prop. xxxvii. The demonstration of this theorem may be made shorter by a reference to the note on Euclid III. Def. 2: for if  $DB$  meet the circle in  $B$  and do not touch it at that point, the line must, when produced, cut the circle in two points.

It is a circumstance worthy of notice, that in this proposition, as well as in Prop. xlviii. Book I. Euclid departs from the ordinary *ex absurdo* mode of proof of converse propositions.

## QUESTIONS ON BOOK III.

1. DEFINE accurately the terms *radius*, *arc*, *circumference*, *chord*, *secant*.
2. How does a *sector* differ in form from a *segment* of a circle? Are they in any case coincident?
3. What is Euclid's criterion of the equality of two circles? What is meant by a given circle? How many points are necessary to determine the *magnitude* and *position* of a circle?
4. When are segments of circles said to be similar? Enunciate the propositions of the Third Book of Euclid, in which this definition is employed. Is it employed in a restricted or general form?
5. In how many points can a circle be cut by a straight line and by another circle?
6. When are straight lines equally distant from the center of a circle?
7. Shew the necessity of an indirect demonstration in Euc. III. 1.
8. Find the centre of a given circle without bisecting any straight line.
9. Shew that if the circumference of one of two equal circles pass through the center of the other, the portions of the two circles, each of which lies without the circumference of the other circle, are equal.
10. If a straight line passing through the center of a circle bisect a straight line in it, it shall cut it at right angles. Point out the exception; and shew that if a straight line bisect the arc and base of a segment of a circle, it will, when produced, pass through the center.

11. If any point be taken within a circle, and a right line be drawn from it to the circumference, how many lines can generally be drawn equal to it? Draw them.

12. Find the shortest distance between a circle and a given straight line without it.

13. Shew that a circle can only have one center, stating the axioms upon which your proof depends.

14. Why would not the demonstration of Euc. III. 9, hold good, if there were only two such equal straight lines?

15. Two parallel chords in a circle are respectively six and eight inches in length, and one inch apart; how many inches is the diameter in length?

16. Which is the greater chord in a circle whose diameter is 10 inches; that whose length is 5 inches, or that whose distance from the center is 4 inches?

17. What is the locus of the middle points of all equal straight lines in a circle?

18. The radius of a circle  $BCDGF$ , (fig. Euc. III. 15.) whose center is  $E$ , is equal to five inches. The distance of the line  $FG$  from the center is four inches, and the distance of the line  $BC$  from the center is three inches, required the lengths of the lines  $FG$ ,  $BC$ .

19. If the chord of an arc be twelve inches long, and be divided into two segments of eight and four inches by another chord: what is the length of the latter chord, if one of its segments be two inches?

20. What is the radius of that circle of which the chords of an arc and of double the arc are five and eight inches respectively?

21. If the chord of an arc of a circle whose diameter is  $8\frac{1}{2}$  inches, be five inches, what is the length of the chord of double the arc of the same circle?

22. State when a straight line is said to touch a circle, and shew from your definition that a straight line cannot be drawn to touch a circle from a point within it.

23. Can more circles than one touch a straight line in the same point?

24. Shew from the construction, Euc. III. 17, that two equal straight lines, and only two, can be drawn touching a given circle from a given point without it: and one, and only one, from a point in the circumference.

25. What is the locus of the centers of all the circles which touch a straight line in a given point?

26. How may a tangent be drawn at a given point in the circumference of a circle, without knowing the center?

27. In a circle place two chords of given length at right angles to each other.

28. From Euc. III. 19, shew how many circles equal to a given circle may be drawn to touch a straight line in the same point.

29. Enunciate Euc. III. 20. Is this true, when the base is greater than a semicircle? If so, why has Euclid omitted this case?

30. The angle at the center of a circle is double of that at the circumference. How will it appear hence that the angle in a semicircle is a right angle?

31. What conditions are essential to the possibility of the inscription and circumscription of a circle in and about a quadrilateral figure?

32. What conditions are requisite in order that a parallelogram may be inscribed in a circle? Are there any analogous conditions requisite that a parallelogram may be described about a circle?

33. Define the angle in a segment of a circle, and the angle on a seg-

ment; and shew that in the same circle, they are together equal to two right angles.

34. State and prove the converse of Euc. III. 22.

35. All circles which pass through two given points have their centers in a certain straight line.

36. Describe the circle of which a given segment is a part. Give Euclid's more simple method of solving the same problem independently of the magnitude of the given segment.

37. In the same circle equal straight lines cut off equal circumferences. If these straight lines have any point common to one another, it must not be in the circumference. Is the enunciation given complete?

38. Enunciate Euc. III. 31, and deduce the proof of it from Euc. III. 20.

39. What is the locus of the vertices of all right-angled triangles which can be described upon the same hypotenuse?

40. How may a perpendicular be drawn to a given straight line from one of its extremities *without producing the line*?

41. If the angle in a semicircle be a right angle; what is the angle in a quadrant?

42. The sum of the squares of any two lines drawn from any point in a semicircle to the extremity of the diameter is constant. Express that *constant* in terms of the radius.

43. In the demonstration of Euc. III. 30, it is stated that "equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less:" explain by reference to the diagram the meaning of this statement.

44. How many circles may be described so as to pass through one, two, and three given points? In what case is it impossible for a circle to pass through three given points?

45. Compare the circumference of the segment (Euc. III. 33.) with the whole circumference when the angle contained in it is a right angle and a half.

46. Include the four cases of Euc. III. 35, in one general proof.

47. Enunciate the propositions which are converse to Props. 32, 35 of Book III.

48. If the position of the center of a circle be known with respect to a given point outside a circle, and the distance of the circumference to the point be ten inches: what is the length of the diameter of the circle, if a tangent drawn from the given point be fifteen inches?

49. If two straight lines be drawn from a point without a circle, and be both terminated by the concave part of the circumference, and if one of the lines pass through the center, and a portion of the other line intercepted by the circle, be equal to the radius: find the diameter of the circle, if the two lines meet the convex part of the circumference,  $a$ ,  $b$ , units respectively from the given point.

50. Upon what propositions depends the demonstration of Euc. III. 35? Is any extension made of this proposition in the Third Book?

51. What conditions must be fulfilled that a circle may pass through four given points?

52. Why is it considered necessary to demonstrate all the separate cases of Euc. III. 35, 36, geometrically, which are comprehended in one formula, when expressed by Algebraic symbols?

53. Enunciate the converse propositions of the Third Book of Euclid which are not demonstrated *ex absurdo*: and state the three methods which Euclid employs in the demonstration of converse propositions in the First and Third Books of the Elements.

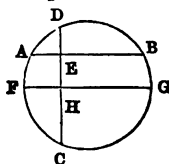


## GEOMETRICAL EXERCISES ON BOOK III.

### PROPOSITION I. THEOREM.

*If AB, CD be chords of a circle at right angles to each other, prove that the sum of the arcs AC, BD is equal to the sum of the arcs AD, BC.*

Draw the diameter  $FGH$  parallel to  $AB$ , and cutting  $CD$  in  $H$ .



Then the arcs  $FDG$  and  $FCG$  are each half the circumference.

Also since  $CD$  is bisected in the point  $H$ ,

the arc  $FD$  is equal to the arc  $FC$ ,

and the arc  $FD$  is equal to the arcs  $FA$ ,  $AD$ , of which,  $AF$  is equal to  $BG$ ,

therefore the arcs  $AD$ ,  $BG$  are equal to the arc  $FC$ ;

add to each  $CG$ ,

therefore the arcs  $AD$ ,  $BC$  are equal to the arcs  $FC$ ,  $CG$ , which make up the half circumference.

Hence also the arcs  $AC$ ,  $DB$  are equal to half the circumference.

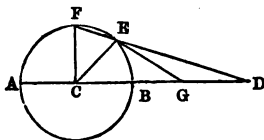
Wherefore the arcs  $AD$ ,  $BC$  are equal to the arcs  $AC$ ,  $DB$ .

### PROPOSITION II. PROBLEM.

*The diameter of a circle having been produced to a given point, it is required to find in the part produced a point, from which if a tangent be drawn to the circle, it shall be equal to the segment of the part produced, that is, between the given point and the point found.*

Analysis. Let  $AEB$  be a circle whose center is  $C$ , and whose diameter  $AB$  is produced to the given point  $D$ .

Suppose that  $G$  is the point required, such that the segment  $GD$  is equal to the tangent  $GE$  drawn from  $G$  to touch the circle in  $E$ .



Join  $DE$  and produce it to meet the circumference again in  $F$ ;  
join also  $CE$  and  $CF$ .

Then in the triangle  $GDE$ , because  $GD$  is equal to  $GE$ ,  
therefore the angle  $GED$  is equal to the angle  $GDE$ ;

and because  $CE$  is equal to  $CF$ ,  
 the angle  $CEF$  is equal to the angle  $CFE$ ;  
 therefore the angles  $CEF$ ,  $GED$  are equal to the angles  $CFE$ ,  
 $GDE$ :

but since  $GE$  is a tangent at  $E$ ,  
 therefore the angle  $CEG$  is a right angle, (III. 18.)  
 hence the angles  $CEF$ ,  $GEF$  are equal to a right angle,  
 and consequently, the angles  $CFE$ ,  $EDG$  are also equal to a right  
 angle,  
 wherefore the remaining angle  $FCD$  of the triangle  $CFD$  is a right  
 angle,

and therefore  $CF$  is perpendicular to  $AD$ .

Synthesis. From the center  $C$ , draw  $CF$  perpendicular to  $AD$   
 meeting the circumference of the circle in  $F$ :

join  $DF$  cutting the circumference in  $E$ ,  
 join also  $CE$ , and at  $E$  draw  $EG$  perpendicular to  $CE$  and inter-  
 secting  $BD$  in  $G$ .

Then  $G$  will be the point required.

For in the triangle  $CFD$ , since  $FCD$  is a right angle, the angles  
 $CFD$ ,  $CDF$  are together equal to a right angle;

also since  $CEG$  is a right angle,  
 therefore the angles  $CEF$ ,  $GED$  are together equal to a right  
 angle;

therefore the angles  $CEF$ ,  $GED$  are equal to the angles  $CFD$ ,  
 $CDF$ ;

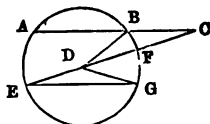
but because  $CE$  is equal to  $CF$ ,  
 the angle  $CEF$  is equal to the angle  $CFD$ ,  
 wherefore the remaining angle  $GED$  is equal to the remaining  
 angle  $CDF$ ,

and the side  $GD$  is equal to the side  $GE$  of the triangle  $EGD$ ,  
 therefore the point  $G$  is determined according to the required  
 conditions.

### PROPOSITION III. THEOREM.

*If a chord of a circle be produced till the part produced be equal to the  
 radius, and if from its extremity a line be drawn through the center and  
 meeting the convex and concave circumferences, the convex is one-third of the  
 concave circumference.*

Let  $AB$  any chord be produced to  $C$ , so that  $BC$  is equal to the  
 radius of the circle:



and let  $CE$  be drawn from  $C$  through the center  $D$ , and meeting  
 the convex circumference in  $F$ , and the concave in  $E$ .

Then the arc  $BF$  is one-third of the arc  $AE$ .

Draw  $EG$  parallel to  $AB$ , and join  $DB, DG$ .

Since the angle  $DEG$  is equal to the angle  $DGE$ ; (I. 5.)

and the angle  $GDF$  is equal to the angles  $DEG, DGE$ ; (I. 32.)

therefore the angle  $GDC$  is double of the angle  $DEG$ .

But the angle  $BDC$  is equal to the angle  $BCD$ , (I. 5.)

and the angle  $CEG$  is equal to the alternate angle  $ACE$ ; (I. 29.)

therefore the angle  $GDC$  is double of the angle  $CDB$ ,

add to these equals the angle  $CDB$ ,

therefore the whole angle  $GDB$  is treble of the angle  $CDB$ ,

but the angles  $GDB, CDB$  at the center  $D$ , are subtended by the arcs  $BF, BG$ , of which  $BG$  is equal to  $AE$ .

Wherefore the circumference  $AE$  is treble of the circumference  $BF$ , and  $BF$  is one-third of  $AE$ .

Hence may be solved the following problem:

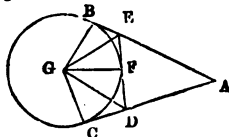
$AE, BF$  are two arcs of a circle intercepted between a chord and a given diameter. Determine the position of the chord, so that one arc shall be triple of the other.

#### PROPOSITION IV. THEOREM.

$AB, AC$  and  $ED$  are tangents to the circle  $CFB$ ; at whatever point between  $C$  and  $B$  the tangent  $EFD$  is drawn, the three sides of the triangle  $AED$  are equal to twice  $AB$  or twice  $AC$ : also the angle subtended by the tangent  $EFD$  at the center of the circle, is a constant quantity.

Take  $G$  the center of the circle, and join  $GB, GE, GF, GD, GC$ .

Then  $EB$  is equal to  $EF$ , and  $DC$  to  $DF$ ; (III. 37.)



therefore  $ED$  is equal to  $EB$  and  $DC$ ;

to each of these add  $AE, AD$ ,

wherefore  $AD, AE, ED$  are equal to  $AB, AC$ ;

and  $AB$  is equal to  $AC$ ,

therefore  $AD, AE, ED$  are equal to twice  $AB$ , or twice  $AC$ ;

or the perimeter of the triangle  $AED$  is a constant quantity.

Again, the angle  $EGF$  is half of the angle  $BGF$ ,

and the angle  $DGF$  is half of the angle  $CGF$ ,

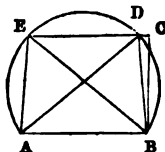
therefore the angle  $DGE$  is half of the angle  $CGB$ ,

or the angle subtended by the tangent  $ED$  at  $G$ , is half of the angle contained between the two radii which meet the circle at the points where the two tangents  $AB, AC$  meet the circle.

#### PROPOSITION V. PROBLEM.

Given the base, the vertical angle, and the perpendicular in a plane triangle, to construct it.

Upon the given base  $AB$  describe a segment of a circle containing an angle equal to the given angle. (III. 33.)



At the point  $B$  draw  $BC$  perpendicular to  $AB$ , and equal to the altitude of the triangle. (I. 11, 8.)

Through  $C$ , draw  $CDE$  parallel to  $AB$ , and meeting the circumference in  $D$  and  $E$ . (I. 31.)

Join  $DA$ ,  $DB$ ; also  $EA$ ,  $EB$ ;

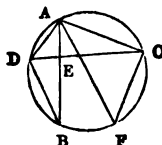
then  $EAB$  or  $DAB$  is the triangle required.

It is also manifest, that if  $CDE$  touch the circle, there will be only one triangle which can be constructed on the base  $AB$  with the given altitude.

#### PROPOSITION VI. THEOREM.

*If two chords of a circle intersect each other at right angles either within or without the circle, the sum of the squares described upon the four segments, is equal to the square described upon the diameter.*

Let the chords  $AB$ ,  $CD$  intersect at right angles in  $E$ .



Draw the diameter  $AF$ , and join  $AC$ ,  $AD$ ,  $CF$ ,  $DB$ .

Then the angle  $ACF$  in a semicircle is a right angle, (III. 31.)

and equal to the angle  $AED$ :

also the angle  $ADC$  is equal to the angle  $AFC$ . (III. 21.)

Hence in the triangles  $ADE$ ,  $AFC$ , there are two angles in the one respectively equal to two angles in the other,

consequently, the third angle  $CAF$  is equal to the third angle  $DAB$ ;

therefore the arc  $DB$  is equal to the arc  $CF$ , (III. 26.)

and therefore also the chord  $DB$  is equal to the chord  $CF$ . (III. 29.)

Because  $AEC$  is a right-angled triangle,

the squares on  $AE$ ,  $EC$  are equal to the square on  $AC$ ; (I. 47.)

similarly, the squares on  $DE$ ,  $EB$  are equal to the square on  $DB$ ;

therefore the squares on  $AE$ ,  $EC$ ,  $DE$ ,  $EB$ , are equal to the squares on  $AC$ ,  $DB$ ;

but  $DB$  was proved equal to  $FC$ ,

and the squares on  $AC$ ,  $FC$  are equal to the square on  $AF$ .

wherefore the squares on  $AE$ ,  $EC$ ,  $DE$ ,  $EB$ , are equal to the square on  $AF$ , the diameter of the circle.

When the chords meet without the circle, the property is proved in a similar manner.

## I.

7. THROUGH a given point within a circle, to draw a chord which shall be bisected in that point, and prove it to be the least.

8. To draw that diameter of a given circle which shall pass at a given distance from a given point.

9. Find the locus of the middle points of any system of parallel chords in a circle.

10. The two straight lines which join the opposite extremities of two parallel chords, intersect in a point in that diameter which is perpendicular to the chords.

11. The straight lines joining towards the same parts, the extremities of any two lines in a circle equally distant from the center, are parallel to each other.

12.  $A, B, C, A', B', C'$  are points on the circumference of a circle; if the lines  $AB, AC$  be respectively parallel to  $A'B', A'C'$ , shew that  $BC$  is parallel to  $B'C'$ .

13. Two chords of a circle being given in position and magnitude, describe the circle.

14. Two circles are drawn, one lying within the other; prove that no chord to the outer circle can be bisected in the point in which it touches the inner, unless the circles are concentric, or the chord be perpendicular to the common diameter. If the circles have the same center, shew that every chord which touches the inner circle is bisected in the point of contact.

15. Draw a chord in a circle, so that it may be double of its perpendicular distance from the center.

16. The arcs intercepted between any two parallel chords in a circle are equal.

17. If any point  $P$  be taken in the plane of a circle, and  $PA, PB, PC, \dots$  be drawn to any number of points  $A, B, C, \dots$  situated symmetrically in the circumference, the sum of  $PA, PB, \dots$  is least when  $P$  is at the center of the circle.

## II.

18. The sum of the arcs subtending the vertical angles made by any two chords that intersect, is the same, as long as the angle of intersection is the same.

19. From a point without a circle two straight lines are drawn cutting the convex and concave circumferences, and also respectively parallel to two radii of the circle. Prove that the difference of the concave and convex arcs intercepted by the cutting lines, is equal to twice the arc intercepted by the radii.

20. In a circle with center  $O$ , any two chords,  $AB, CD$  are drawn

cutting in  $E$ , and  $OA, OB, OC, OD$  are joined; prove that the angles  $\angle AOC + \angle BOD = 2 \cdot \angle AEC$ , and  $\angle AOD + \angle BOC = 2 \cdot \angle AED$ .

21. If from any point without a circle, lines be drawn cutting the circle and making equal angles with the longest line, they will cut off equal segments.

22. If the corresponding extremities of two intersecting chords of a circle be joined, the triangles thus formed will be equiangular.

23. Through a given point within or without a circle, it is required to draw a straight line cutting off a segment containing a given angle.

24. If on two lines containing an angle, segments of circles be described containing angles equal to it, the lines produced will touch the segments.

25. Any segment of a circle being described on the base of a triangle; to describe on the other sides segments similar to that on the base.

26. If an arc of a circle be divided into three equal parts by three straight lines drawn from one extremity of the arc, the angle contained by two of the straight lines is bisected by the third.

27. If the chord of a given circular segment be produced to a fixed point, describe upon it when so produced a segment of a circle which shall be similar to the given segment, and shew that the two segments have a common tangent.

28. If  $AD, CE$  be drawn perpendicular to the sides  $BC, AB$  of the triangle  $ABC$ , and  $DE$  be joined, prove that the angles  $\angle ADE$ , and  $\angle ACE$  are equal to each other.

29. If from any point in a circular arc, perpendiculars be let fall on its bounding radii, the distance of their feet is invariable.

### III.

30. If both tangents be drawn, (fig. Euc. III. 17.) and the points of contact joined by a straight line which cuts  $EA$  in  $H$ , and on  $HA$  as diameter a circle be described, the lines drawn through  $E$  to touch this circle will meet it on the circumference of the given circle.

31. Draw, (1) perpendicular, (2) parallel to a given line, a line touching a given circle.

32. If two straight lines intersect, the centers of all circles that can be inscribed between them, lie in two lines at right angles to each other.

33. Draw two tangents to a given circle, which shall contain an angle equal to a given rectilineal angle.

34. Describe a circle with a given radius touching a given line, and so that the tangents drawn to it from two given points in this line may be parallel, and shew that if the radius vary, the locus of the centers of the circles so described is a circle.

35. Determine the distance of a point from the center of a given circle, so that if tangents be drawn from it to the circle, the concave part of the circumference may be double of the convex.

36. In a chord of a circle produced, it is required to find a point, from which if a straight line be drawn touching the circle, the line so drawn shall be equal to a given straight line.

37. Find a point without a given circle, such that the sum of the two lines drawn from it touching the circle, shall be equal to the line drawn from it through the center to meet the circle.

38. If from a point without a circle two tangents be drawn; the straight line which joins the points of contact will be bisected at right angles by a line drawn from the center to the point without the circle.

39. If tangents be drawn at the extremities of any two diameters of a circle, and produced to intersect one another; the straight lines joining the opposite points of intersection will both pass through the center.

40. If from any point without a circle two lines be drawn touching the circle, and from the extremities of any diameter, lines be drawn to the point of contact cutting each other within the circle, the line drawn from the points without the circle to the point of intersection, shall be perpendicular to the diameter.

41. If any chord of a circle be produced equally both ways, and tangents to the circle be drawn on opposite sides of it from its extremities, the line joining the points of contact bisects the given chord.

42.  $AB$  is a chord, and  $AD$  is a tangent to a circle at  $A$ .  $DPQ$  any secant parallel to  $AB$  meeting the circle in  $P$  and  $Q$ . Shew that the triangle  $PAD$  is equiangular with the triangle  $QAB$ .

43. If from any point in the circumference of a circle a chord and tangent be drawn, the perpendiculars dropped upon them from the middle point of the subtended arc, are equal to one another.

#### IV.

44. In a given straight line to find a point at which two other straight lines being drawn to two given points, shall contain a right angle. Shew that if the distance between the two given points be *greater* than the sum of their distances from the given line, there will be two such points; if *equal*, there may be only one; if *less*, the problem may be impossible.

45. Find the point in a given straight line at which the tangents to a given circle will contain the greatest angle.

46. Of all straight lines which can be drawn from two given points to meet in the convex circumference of a given circle, the sum of those two will be the least, which make equal angles with the tangent at the point of concurrence.

47.  $DF$  is a straight line touching a circle, and terminated by  $AD, BF$ , the tangents at the extremities of the diameter  $AB$ , shew that the angle which  $DF$  subtends at the center is a right angle.

48. If tangents  $Am, Bn$  be drawn at the extremities of the diameter of a semicircle, and any line in  $mPn$  crossing them and touching the circle in  $P$ , and if  $AN, BM$  be joined intersecting in  $O$  and cutting the semicircle in  $E$  and  $F$ ; shew that  $O, P$ , and the point of intersection of the tangents at  $E$  and  $F$ , are in the same straight line.

49. If from a point  $P$  without a circle, any straight line be drawn cutting the circumference in  $A$  and  $B$ , shew that the straight lines joining the points  $A$  and  $B$  with the bisection of the chord of contact of the tangents from  $P$ , make equal angles with that chord.

## V.

50. Describe a circle which shall pass through a given point and which shall touch a given straight line in a given point.

51. Draw a straight line which shall touch a given circle, and make a given angle with a given straight line.

52. Describe a circle the circumference of which shall pass through a given point and touch a given circle in a given point.

53. Describe a circle with a given center, such that the circle so described and a given circle may touch one another internally.

54. Describe the circles which shall pass through a given point and touch two given straight lines.

55. Describe a circle with a given center, cutting a given circle in the extremities of a diameter.

56. Describe a circle which shall have its center in a given straight line, touch another given line, and pass through a fixed point in the first given line.

57. The center of a given circle is equidistant from two given straight lines; to describe another circle which shall touch the two straight lines and shall cut off from the given circle a segment containing an angle equal to a given rectilineal angle.

## VI.

58. If any two circles, the centers of which are given, intersect each other, the greatest line which can be drawn through either point of intersection and terminated by the circles, is independent of the diameters of the circles.

59. Two equal circles intersect, the lines joining the points in which any straight line through one of the points of section, which meets the circles with the other point of section, are equal.

60. Draw through one of the points in which any two circles cut one another, a straight line which shall be terminated by their circumferences and bisected in their point of section.

61. Describe two circles with given radii which shall cut each other, and have the line between the points of section equal to a given line.

62. Two circles cut each other, and from the points of intersection straight lines are drawn parallel to one another, the portions intercepted by the circumferences are equal.

63.  $ACB$ ,  $ADB$  are two segments of circles on the same base  $AB$ , take any point  $C$  in the segment  $ACB$ ; join  $AC$ ,  $BC$ , and produce them to meet the segment  $ADB$  in  $D$  and  $E$  respectively: shew that the arc  $DE$  is constant.

64.  $ADB$ ,  $ACB$ , are the arcs of two equal circles cutting one another in the straight line  $AB$ , draw the chord  $ACD$  cutting the inner circumference in  $C$  and the outer in  $D$ , such that  $AD$  and  $DB$  together may be double of  $AC$  and  $CB$  together.

65. If from two fixed points in the circumference of a circle, straight lines be drawn intercepting a given arc and meeting without the circle, the locus of their intersections is a circle.



66. If two circles intersect, the common chord produced bisects the common tangent.

67. Shew that, if two circles cut each other, and from any point in the straight line produced, which joins their intersections, two tangents be drawn, one to each circle, they shall be equal to one another.

68. Two circles intersect in the points  $A$  and  $B$ ; through  $A$  and  $B$  any two straight lines  $CAD$ ,  $EBF$ , are drawn cutting the circles in the points  $C$ ,  $D$ ,  $E$ ,  $F$ ; prove that  $CE$  is parallel to  $DF$ .

69. Two equal circles are drawn intersecting in the points  $A$  and  $B$ , a third circle is drawn with center  $A$  and any radius not greater than  $AB$  intersecting the former circles in  $D$  and  $C$ . Shew that the three points,  $B$ ,  $C$ ,  $D$  lie in one and the same straight line.

70. If two circles cut each other, the straight line joining their centers will bisect their common chord at right angles.

71. Two circles cut one another; if through a point of intersection a straight line is drawn bisecting the angle between the diameters at that point, this line cuts off similar segments in the two circles.

72.  $ACB$ ,  $APB$  are two equal circles, the center of  $APB$  being on the circumference of  $ACB$ ,  $AB$  being the common chord, if any chord  $AC$  of  $ACB$  be produced to cut  $APB$  in  $P$ , the triangle  $PBC$  is equilateral.

## VII.

73. If two circles touch each other externally, and two parallel lines be drawn, so touching the circles in points  $A$  and  $B$  respectively that neither circle is cut, then a straight line  $AB$  will pass through the point of contact of the circles.

74. A common tangent is drawn to two circles which touch each other externally; if a circle be described on that part of it which lies between the points of contact, as diameter, this circle will pass through the point of contact of the two circles, and will touch the line which joins their centers.

75. If two circles touch each other externally or internally, and parallel diameters be drawn, the straight line joining the extremities of these diameters will pass through the point of contact.

76. If two circles touch each other internally, and any circle be described touching both, prove that the sum of the distances of its center from the centers of the two given circles will be invariable.

77. If two circles touch each other, any straight line passing through the point of contact, cuts off similar parts of their circumferences.

78. Two circles touch each other externally, the diameter of one being double of the diameter of the other; through the point of contact any line is drawn to meet the circumferences of both; shew that the part of the line which lies in the larger circle is double of that in the smaller.

79. If a circle roll within another of twice its size, any point in its circumference will trace out a diameter of the first.

80. With a given radius, to describe a circle touching two given circles.

81. Two equal circles touch one another externally, and through the point of contact chords are drawn, one to each circle, at right angles to each; prove that the straight line joining the other extremities of these chords is equal and parallel to the straight line joining the centres of the circles.

82. Two circles can be described, each of which shall touch a given circle, and pass through two given points outside the circle; shew that the angles which the two given points subtend at the two points of contact, are one greater and the other less than that which they subtend at any other point in the given circle.

## VIII.

83. Draw a straight line which shall touch two given circles; (1) on the same side; (2) on the alternate sides.

84. If two circles do not touch each other, and a segment of the line joining their centers be intercepted between the convex circumferences, any circle whose diameter is not less than that segment may be so placed as to touch both the circles.

85. Given two circles: it is required to find a point from which tangents may be drawn to each, equal to two given straight lines.

86. Two circles are traced on a plane; draw a straight line cutting them in such a manner that the chords intercepted within the circles shall have given lengths.

87. Draw a straight line which shall touch one of two given circles and cut off a given segment from the other. Of how many solutions does this problem admit?

88. If from the point where a common tangent to two circles meets the line joining their centers, any line be drawn cutting the circles, it will cut off similar segments.

89. To find a point  $P$ , so that tangents drawn from it to the out-sides of two equal circles which touch each other, may contain an angle equal to a given angle.

90. Describe a circle which shall touch a given straight line at a given point, and bisect the circumference of a given circle.

91. A circle is described to pass through a given point and cut a given circle orthogonally, shew that the locus of the center is a certain straight line.

92. Through two given points to describe a circle bisecting the circumference of a given circle.

93. Describe a circle through a given point, and touching a given straight line, so that the chord joining the given point and point of contact, may cut off a segment containing a given angle.

94. To describe a circle through two given points to cut a straight line given in position, so that a diameter of the circle drawn through the point of intersection, shall make a given angle with the line.

95. Describe a circle which shall pass through two given points and cut a given circle, so that the chord of intersection may be of a given length.

## IX.

96. The circumference of one circle is wholly within that of another. Find the greatest and the least straight lines that can be drawn touching the former and terminated by the latter.

97. Draw a straight line through two concentric circles, so that the chord terminated by the exterior circumference may be double that terminated by the interior. What is the least value of the radius of the interior circle for which the problem is possible?

98. If a straight line be drawn cutting any number of concentric circles, shew that the segments so cut off are not similar.

99. If from any point in the circumference of the exterior of two concentric circles, two straight lines be drawn touching the interior and meeting the exterior; the distance between the points of contact will be half that between the points of intersection.

100. Shew that all equal straight lines in a circle will be touched by another circle.

101. Through a given point draw a straight line so that the part intercepted by the circumference of a circle, shall be equal to a given straight line not greater than the diameter.

102. Two circles are described about the same center, draw a chord to the outer circle, which shall be divided into three equal parts by the inner one. How is the possibility of the problem limited?

103. Find a point without a given circle from which if two tangents be drawn to it, they shall contain an angle equal to a given angle, and shew that the locus of this point is a circle concentric with the given circle.

104. Draw two concentric circles such that those chords of the outer circle which touch the inner, may be equal to its diameter.

105. Find a point in a given straight line from which the tangent drawn to a given circle, is of given length.

106. If any number of chords be drawn in the inner of two concentric circles, from the same point  $A$  in its circumference, and each of the chords be then produced beyond  $A$  to the circumference of the outer circle, the rectangle contained by the whole line so produced and the part of it produced, shall be constant for all the cases.

## X.

107. The circles described on the sides of any triangle as diameters will intersect in the sides, or sides produced, of the triangle.

108. The circles which are described upon the sides of a right-angled triangle as diameters, meet the hypotenuse in the same point; and the line drawn from the point of intersection to the center of either of the circles will be a tangent to the other circle.

109. If on the sides of a triangle circular arcs be described containing angles whose sum is equal to two right angles, the triangle formed by the lines joining their centers, has its angles equal to those in the segments.

110. The perpendiculars let fall from the three angles of any triangle upon the opposite sides, intersect each other in the same point.

111. If  $AD$ ,  $CE$  be drawn perpendicular to the sides  $BC$ ,  $AB$  of

the triangle  $ABC$ , prove that the rectangle contained by  $BC$  and  $BD$ , is equal to the rectangle contained by  $BA$  and  $BE$ .

112. The lines which bisect the vertical angles of all triangles on the same base and with the same vertical angle, all intersect in one point.

113. Of all triangles on the same base and between the same parallels, the isosceles has the greatest vertical angle.

114. It is required within an isosceles triangle to find a point such, that its distance from one of the equal angles may be double its distance from the vertical angle.

115. To find within an acute-angled triangle, a point from which, if straight lines be drawn to the three angles of the triangle, they shall make equal angles with each other.

116. A flag-staff of a given height is erected on a tower whose height is also given: at what point on the horizon will the flag-staff appear under the greatest possible angle?

117. A ladder is gradually raised against a wall; find the locus of its middle point.

118. The triangle formed by the chord of a circle (produced or not), the tangent at its extremity, and any line perpendicular to the diameter through its other extremity, will be isosceles.

119.  $AD$ ,  $BE$  are perpendiculars from the angles  $A$  and  $B$  on the opposite sides of a triangle,  $BF$  perpendicular to  $ED$  or  $ED$  produced; shew that the angle  $FBD = EBA$ .

#### XI.

120. If three equal circles have a common point of intersection, prove that a straight line joining any two of the points of intersection, will be perpendicular to the straight line joining the other two points of intersection.

121. Two equal circles cut one another, and a third circle touches each of these two equal circles externally; the straight line which joins the points of section will, if produced, pass through the center of the third circle.

122. A number of circles touch each other at the same point, and a straight line is drawn from it cutting them: the straight lines joining each point of intersection with the center of the circle will be all parallel.

123. If three circles intersect one another, two and two, the three chords joining the points of intersection shall all pass through one point.

124. If three circles touch each other externally, and the three common tangents be drawn, these tangents shall intersect in a point equidistant from the points of contact of the circles.

125. If two equal circles intersect one another in  $A$  and  $B$ , and from one of the points of intersection as a center, a circle be described which shall cut both of the equal circles, then will the other point of intersection, and the two points in which the third circle cuts the other two on the same side of  $AB$ , be in the same straight line.

#### XII.

126. Given the base, the vertical angle, and the difference of the sides, to construct the triangle.

127. Describe a triangle, having given the vertical angle, and the segments of the base made by a line bisecting the vertical angle.

128. Given the perpendicular height, the vertical angle and the sum of the sides, to construct the triangle.

129. Construct a triangle in which the vertical angle and the difference of the two angles at the base shall be respectively equal to two given angles, and whose base shall be equal to a given straight line.

130. Given the vertical angle, the difference of the two sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex; construct the triangle.

131. Given the vertical angle, and the lengths of two lines drawn from the extremities of the base to the points of bisection of the sides, to construct the triangle.

132. Given the base, and vertical angle, to find the triangle whose area is a maximum.

133. Given the base, the altitude, and the sum of the two remaining sides; construct the triangle.

134. Describe a triangle of given base, area, and vertical angle.

135. Given the base and vertical angle of a triangle, find the locus of the intersection of perpendiculars to the sides from the extremities of the base.

### XIII.

136. Shew that the perpendiculars to the sides of a quadrilateral inscribed in a circle from their middle points intersect in a fixed point.

137. The lines bisecting any angle of a quadrilateral figure inscribed in a circle, and the opposite exterior angle, meet in the circumference of the circle.

138. If two opposite sides of a quadrilateral figure inscribed in a circle be equal, prove that the other two are parallel.

139. The angles subtended at the center of a circle by any two opposite sides of a quadrilateral figure circumscribed about it, are together equal to two right angles.

140. Four circles are described so that each may touch internally three of the sides of a quadrilateral figure, or one side and the adjacent sides produced; shew that the centers of these four circles will all lie in the circumference of a circle.

141. One side of a trapezium capable of being inscribed in a given circle is given, the sum of the remaining three sides is given; and also one of the angles opposite to the given side: construct it.

142. If the sides of a quadrilateral figure inscribed in a circle be produced to meet, and from each of the points of intersection a straight line be drawn, touching the circle, the squares on these tangents are together equal to the square on the straight line joining the points of intersection.

143. If a quadrilateral figure be described about a circle, the sums of the opposite sides are equal; and each sum equal to half the perimeter of the figure.

144. A quadrilateral  $ABCD$  is inscribed in a circle,  $BC$  and  $DC$

are produced to meet  $AD$  and  $AB$  produced in  $E$  and  $F$ . The angles  $ABC$  and  $ADC$  are together equal to  $AFC$ ,  $AEB$ , and twice the angle  $BAC$ .

145. If the hypotenuse  $AB$  of a right-angled triangle  $ABC$  be bisected in  $D$ , and  $EDF$  drawn perpendicular to  $AB$ , and  $DE$ ,  $DF$  cut off each equal to  $DA$ , and  $CE$ ,  $CF$  joined, prove that the last two lines will bisect the angle at  $C$  and its supplement respectively.

146.  $ABCD$  is a quadrilateral figure inscribed in a circle. Through its angular points tangents are drawn so as to form another quadrilateral figure  $FBLCHDEA$  circumscribed about the circle. Find the relation which exists between the angles of the exterior and the angles of the interior figure.

147. The angle contained by the tangents drawn at the extremities of any chord in a circle is equal to the difference of the angles in segments made by the chord: and also equal to twice the angle contained by the same chord and a diameter drawn from either of its extremities.

148. If  $ABCD$  be a quadrilateral figure, and the lines  $AB$ ,  $AC$ ,  $AD$  be equal, shew that the angle  $BAD$  is double of  $CBD$  and  $CDB$  together.

149. Shew that the four lines which bisect the interior angles of a quadrilateral figure, form by their intersections, a quadrilateral figure which can be inscribed in a circle.

150. In a quadrilateral figure  $ABCD$  is inscribed a second quadrilateral by joining the middle points of its adjacent sides; a third is similarly inscribed in the second, and so on. Shew that each of the series of quadrilaterals will be capable of being inscribed in a circle if the first three are so. Shew also that two at least of the opposite sides of  $ABCD$  must be equal, and that the two squares upon these sides are together equal to the sum of the squares upon the other two.

#### XIV.

151. If from any point in the diameter of a semicircle, there be drawn two straight lines to the circumference, one to the bisection of the circumference, the other at right angles to the diameter, the squares upon these two lines are together double of the square upon the semi-diameter.

152. If from any point in the diameter of a circle, straight lines be drawn to the extremities of a parallel chord, the squares on these lines are together equal to the squares on the segments into which the diameter is divided.

153. From a given point without a circle, at a distance from the circumference of the circle not greater than its diameter, draw a straight line to the concave circumference which shall be bisected by the convex circumference.

154. If any two chords be drawn in a circle perpendicular to each other, the sum of their squares is equal to twice the square on the diameter diminished by four times the square on the line joining the center with their point of intersection.

155. Two points are taken in the diameter of a circle at any equal distances from the center; through one of these draw any chord, and join its extremities and the other point. The triangle so formed has the sum of the squares of its sides invariable.

156. If chords drawn from any fixed point in the circumference of a circle, be cut by another chord which is parallel to the tangent at that point, the rectangle contained by each chord, and the part of it intercepted between the given point and the given chord, is constant.

157. If  $AB$  be a chord of a circle inclined by half a right angle to the tangent at  $A$ , and  $AC$ ,  $AD$  be any two chords equally inclined to  $AB$ ,  $AC^2 + AD^2 = 2 \cdot AB^2$ .

158. A chord  $POQ$  cuts the diameter of a circle in  $Q$ , in an angle equal to half a right angle;  $PO^2 + OQ^2 = 2(\text{rad.})^2$ .

159. Let  $ACDB$  be a semicircle whose diameter is  $AB$ ; and  $AD$ ,  $BC$  any two chords intersecting in  $P$ ; prove that

$$AB^2 = DA \cdot AP + CB \cdot BP.$$

160. If  $ABDC$  be any parallelogram, and if a circle be described passing through the point  $A$ , and cutting the sides  $AB$ ,  $AC$ , and the diagonal  $AD$ , in the points  $F$ ,  $G$ ,  $H$  respectively, shew that

$$AB \cdot AF + AC \cdot AG = AD \cdot AH.$$

161. Produce a given straight line, so that the rectangle under the given line, and the whole line produced, may equal the square of the part produced.

162. If  $A$  be a point within a circle,  $BC$  the diameter, and through  $A$ ,  $AD$  be drawn perpendicular to the diameter, and  $BAE$  meeting the circumference in  $E$ , then  $BA \cdot BE = BC \cdot BD$ .

163. The diameter  $ACD$  of a circle, whose center is  $C$ , is produced to  $P$ , determine a point  $F$  in the line  $AP$  such that the rectangle  $PF \cdot PC$  may be equal to the rectangle  $PD \cdot PA$ .

164. To produce a given straight line, so that the rectangle contained by the whole line thus produced, and the part of it produced, shall be equal to a given square.

165. Two straight lines stand at right angles to each other, one of which passes through the center of a given circle, and from any point in the other, tangents are drawn to the circle. Prove that the chord joining the points of contact cuts the first line in the same point, whatever be the point in the second from which the tangents are drawn.

166.  $A$ ,  $B$ ,  $C$ ,  $D$ , are four points in order in a straight line, find a point  $E$  between  $B$  and  $C$ , such that  $AE \cdot EB = ED \cdot EC$ , by a geometrical construction.

167. If any two circles touch each other in the point  $O$ , and lines be drawn through  $O$  at right angles to each other, the one line cutting the circles in  $P$ ,  $P'$ , the other in  $Q$ ,  $Q'$ ; and if the line joining the centers of the circles cut them in  $A$ ,  $A'$ ; then

$$PP' + QQ' = AA'^2.$$

## BOOK IV.

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### DEFINITIONS.

#### I.

A RECTILINEAL figure is said to be inscribed in another rectilinear figure, when all the angular points of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.



#### II.

In like manner, a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

#### III.

A rectilinear figure is said to be inscribed in a circle, when all the angular points of the inscribed figure are upon the circumference of the circle.



#### IV.

A rectilinear figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.



#### V.

In like manner, a circle is said to be inscribed in a rectilinear figure, when the circumference of the circle touches each side of the figure.

#### VI.

A circle is said to be described about a rectilinear figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.





## VII.

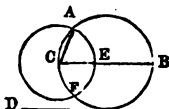
A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

## PROPOSITION I. PROBLEM.

*In a given circle to place a straight line, equal to a given straight line which is not greater than the diameter of the circle.*

Let  $ABC$  be the given circle, and  $D$  the given straight line, not greater than the diameter of the circle.

It is required to place in the circle  $ABC$  a straight line equal to  $D$ .



Draw  $BC$  the diameter of the circle  $ABC$ .

Then, if  $BC$  is equal to  $D$ , the thing required is done; for in the circle  $ABC$  a straight line  $BC$  is placed equal to  $D$ .

But, if it is not,  $BC$  is greater than  $D$ ; (hyp.)

make  $CE$  equal to  $D$ , (I. 3.)

and from the center  $C$ , at the distance  $CE$ , describe the circle  $AEF$ , and join  $CA$ .

Then  $CA$  shall be equal to  $D$ .

Because  $C$  is the center of the circle  $AEF$ ,

therefore  $CA$  is equal to  $CE$ : (I. def. 15.)

but  $CE$  is equal to  $D$ ; (constr.)

therefore  $D$  is equal to  $CA$ . (ax. 1.)

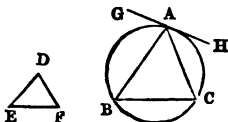
Wherefore in the circle  $ABC$ , a straight line  $CA$  is placed equal to the given straight line  $D$ , which is not greater than the diameter of the circle. Q.E.F.

## PROPOSITION II. PROBLEM.

*In a given circle to inscribe a triangle equiangular to a given triangle.*

Let  $ABC$  be the given circle, and  $DEF$  the given triangle.

It is required to inscribe in the circle  $ABC$  a triangle equiangular to the triangle  $DEF$ .



Draw the straight line  $GAH$  touching the circle in the point  $A$ , (III. 17.) and at the point  $A$ , in the straight line  $AH$ ,

make the angle  $HAC$  equal to the angle  $DEF$ ; (I. 23.)

and at the point  $A$ , in the straight line  $AG$ ,

make the angle  $GAB$  equal to the angle  $DFE$ ;

and join  $BC$ : then  $ABC$  shall be the triangle required.

Because  $HAG$  touches the circle  $ABC$ ,

and  $AC$  is drawn from the point of contact,

therefore the angle  $HAC$  is equal to the angle  $ABC$  in the alternate segment of the circle: (III. 32.)

but  $HAC$  is equal to the angle  $DEF$ ; (constr.)

therefore also the angle  $ABC$  is equal to  $DEF$ : (ax. 1.)

for the same reason, the angle  $ACB$  is equal to the angle  $DFE$ ;

therefore the remaining angle  $BAC$  is equal to the remaining angle  $EDF$ : (I. 32. and ax. 3.)

wherefore the triangle  $ABC$  is equiangular to the triangle  $DEF$ ,

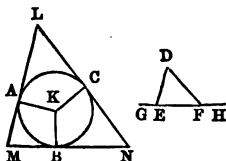
and it is inscribed in the circle  $ABC$ . Q.E.F.

### PROPOSITION III. PROBLEM.

*About a given circle to describe a triangle equiangular to a given triangle.*

Let  $ABC$  be the given circle, and  $DEF$  the given triangle.

It is required to describe a triangle about the circle  $ABC$  equiangular to the triangle  $DEF$ .



Produce  $EF$  both ways to the points  $G, H$ ;

find the center  $K$  of the circle  $ABC$ , (III. 1.)

and from it draw any straight line  $KB$ ;

at the point  $K$  in the straight line  $KB$ ,

make the angle  $BKA$  equal to the angle  $DEG$ , (I. 23.)

and the angle  $BKC$  equal to the angle  $DFH$ ;

and through the points  $A, B, C$ , draw the straight lines  $LAM, MBN, NCL$ , touching the circle  $ABC$ . (III. 17.)

Then  $LMN$  shall be the triangle required.

Because  $LM, MN, NL$  touch the circle  $ABC$  in the points  $A, B, C$ , to which from the center are drawn  $KA, KB, KC$ ,

therefore the angles at the points  $A, B, C$  are right angles: (III. 18.)

and because the four angles of the quadrilateral figure  $AMBK$  are equal to four right angles,

for it can be divided into two triangles;

and that two of them  $KAM, KBM$  are right angles,

therefore the other two  $AKB, AMB$  are equal to two right angles:

(ax. 3.)

but the angles  $DEG, DEF$  are likewise equal to two right angles;

(I. 13.)

therefore the angles  $AKB$ ,  $AMB$  are equal to the angles  $DEG$ ,  $DEF$ ;  
(ax. 1.)

of which  $AKB$  is equal to  $DEG$ ; (constr.)  
wherefore the remaining angle  $AMB$  is equal to the remaining angle  
 $DEF$ . (ax. 3.)

In like manner, the angle  $LNM$  may be demonstrated to be equal  
to  $DFE$ ;

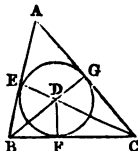
and therefore the remaining angle  $MLN$  is equal to the remaining  
angle  $EDF$ : (I. 32 and ax. 3.)

therefore the triangle  $LMN$  is equiangular to the triangle  $DEF$ :  
and it is described about the circle  $ABC$ . Q.E.F.

#### PROPOSITION IV. PROBLEM.

*To inscribe a circle in a given triangle.*

Let the given triangle be  $ABC$ .  
It is required to inscribe a circle in  $ABC$



Bisect the angles  $ABC$ ,  $BCA$  by the straight lines  $BD$ ,  $CD$  meeting  
one another in the point  $D$ , (I. 9.)

from which draw  $DE$ ,  $DF$ ,  $DG$  perpendiculars to  $AB$ ,  $BC$ ,  $CA$ . (I. 12.)

And because the angle  $EBD$  is equal to the angle  $FBD$ ,

for the angle  $ABC$  is bisected by  $BD$ ,

and that the right angle  $BED$  is equal to the right angle  $BFD$ ; (ax. 11.)

therefore the two triangles  $EBD$ ,  $FBD$  have two angles of the one  
equal to two angles of the other, each to each;

and the side  $BD$ , which is opposite to one of the equal angles in each,  
is common to both;

therefore their other sides are equal; (I. 26.)

wherefore  $DE$  is equal to  $DF$ :

for the same reason,  $DG$  is equal to  $DF$ :

therefore  $DE$  is equal to  $DG$ : (ax. 1.)

therefore the three straight lines  $DE$ ,  $DF$ ,  $DG$  are equal to one  
another;

and the circle described from the center  $D$ , at the distance of any  
of them, will pass through the extremities of the other two, and  
touch the straight lines  $AB$ ,  $BC$ ,  $CA$ ,

because the angles at the points  $E$ ,  $F$ ,  $G$  are right angles,  
and the straight line which is drawn from the extremity of a diameter  
at right angles to it, touches the circle: (III. 16.)

therefore the straight lines  $AB$ ,  $BC$ ,  $CA$  do each of them touch the  
circle,

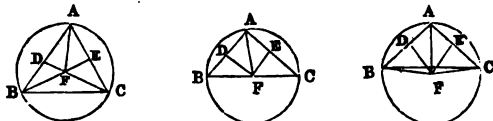
and therefore the circle  $EFG$  is inscribed in the triangle  $ABC$ . Q.E.F.

**PROPOSITION V. PROBLEM.**

*To describe a circle about a given triangle.*

Let the given triangle be  $ABC$ .

It is required to describe a circle about  $ABC$ .



Bisect  $AB$ ,  $AC$  in the points  $D$ ,  $E$ , (I. 10.)  
and from these points draw  $DF$ ,  $EF$  at right angles to  $AB$ ,  $AC$ ; (I. 11.)

$DF$ ,  $EF$  produced meet one another:

for, if they do not meet, they are parallel,

wherefore  $AB$ ,  $AC$ , which are at right angles to them, are parallel;  
which is absurd:

let them meet in  $F$ , and join  $FA$ ;

also, if the point  $F$  be not in  $BC$ , join  $BF$ ,  $CF$ .

Then, because  $AD$  is equal to  $DB$ , and  $DF$  common, and at right angles to  $AB$ ,

therefore the base  $AF$  is equal to the base  $FB$ . (I. 4.)

In like manner, it may be shewn that  $CF$  is equal to  $FA$ ;

and therefore  $BF$  is equal to  $FC$ ; (ax. 1.)

and  $FA$ ,  $FB$ ,  $FC$  are equal to one another:

wherefore the circle described from the center  $F$ , at the distance of one of them, will pass through the extremities of the other two, and be described about the triangle  $ABC$ . Q.E.F.

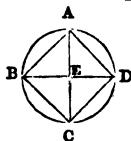
**COR.**—And it is manifest, that when the center of the circle falls within the triangle, each of its angles is less than a right angle, (III. 31.) each of them being in a segment greater than a semicircle; but, when the center is in one of the sides of the triangle, the angle opposite to this side, being in a semicircle, (III. 31.) is a right angle; and, if the center falls without the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, (III. 31.) is greater than a right angle: therefore, conversely, if the given triangle be acute-angled, the center of the circle falls within it; if it be a right-angled triangle, the center is in the side opposite to the right angle; and if it be an obtuse-angled triangle, the center falls without the triangle, beyond the side opposite to the obtuse angle.

**PROPOSITION VI. PROBLEM.**

*To inscribe a square in a given circle.*

Let  $ABCD$  be the given circle.

It is required to inscribe a square in  $ABCD$ .



Draw the diameters,  $AC$ ,  $BD$ , at right angles to one another, (III. 1. and I. 11.)

and join  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ .

The figure  $ABCD$  shall be the square required.

Because  $BE$  is equal to  $ED$ , for  $E$  is the center, and that  $EA$  is common, and at right angles to  $BD$ ;

the base  $BA$  is equal to the base  $AD$ : (I. 4.)

and, for the same reason,  $BC$ ,  $CD$  are each of them equal to  $BA$ , or  $AD$ ;

therefore the quadrilateral figure  $ABCD$  is equilateral.

It is also rectangular;

for the straight line  $BD$  being the diameter of the circle  $ABCD$ .

$BAD$  is a semicircle;

wherefore the angle  $BAD$  is a right angle: (III. 31.)

for the same reason, each of the angles  $ABC$ ,  $BCD$ ,  $CDA$  is a right angle:

therefore the quadrilateral figure  $ABCD$  is rectangular:

and it has been shewn to be equilateral,

therefore it is a square: (I. def. 30.)

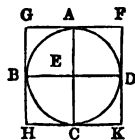
and it is inscribed in the circle  $ABCD$ . Q.E.F.

#### PROPOSITION VII. PROBLEM.

*To describe a square about a given circle.*

Let  $ABCD$  be the given circle.

It is required to describe a square about it.



Draw two diameters  $AC$ ,  $BD$  of the circle  $ABCD$ , at right angles to one another, and through the points  $A$ ,  $B$ ,  $C$ ,  $D$ , draw  $FG$ ,  $GH$ ,  $HK$ ,  $KF$  touching the circle. (III. 17.)

The figure  $GHKF$  shall be the square required.

Because  $FG$  touches the circle  $ABCD$ , and  $EA$  is drawn from the center  $E$  to the point of contact  $A$ ,

therefore the angles at  $A$  are right angles: (III. 18.)

for the same reason, the angles at the points  $B$ ,  $C$ ,  $D$  are right angles;

and because the angle  $AEB$  is a right angle, as likewise is  $EBG$ ,

therefore  $GH$  is parallel to  $AC$ : (I. 28.)

for the same reason  $AC$  is parallel to  $FK$ :

and in like manner  $GF$ ,  $HK$  may each of them be demonstrated to be parallel to  $BD$ :

therefore the figures  $GK$ ,  $GC$ ,  $AK$ ,  $FB$ ,  $BK$  are parallelograms;

and therefore  $GF$  is equal to  $HK$ , and  $GH$  to  $FK$ : (I. 34.)

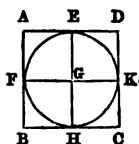
and because  $AC$  is equal to  $BD$ , and that  $AC$  is equal to each of the two  $GH$ ,  $FK$ ;

and  $BD$  to each of the two  $GF, HK$ :  
 $GH, FK$  are each of them equal to  $GF$ , or  $HK$ ;  
 therefore the quadrilateral figure  $FGHK$  is equilateral.  
 It is also rectangular;  
 for  $GBEA$  being a parallelogram, and  $AEB$  a right angle,  
 therefore  $AGB$  is likewise a right angle: (I. 34.)  
 and in the same manner it may be shewn that the angles at  $H, K, F$   
 are right angles:  
 therefore the quadrilateral figure  $FGHK$  is rectangular:  
 and it was demonstrated to be equilateral;  
 therefore it is a square; (I. def. 30.)  
 and it is described about the circle  $ABCD$ . Q.E.F.

## PROPOSITION VIII. PROBLEM.

*To inscribe a circle in a given square.*

Let  $ABCD$  be the given square.  
 It is required to inscribe a circle in  $ABCD$ .



Bisect each of the sides  $AB, AD$  in the points  $F, E$ , (I. 10.)  
 and through  $E$  draw  $EH$  parallel to  $AB$  or  $DC$ , (I. 31.)  
 and through  $F$  draw  $FK$  parallel to  $AD$  or  $BC$ :  
 therefore each of the figures  $AK, KB, AH, HD, AG, GC, BG, GD$   
 is a right-angled parallelogram;  
 and their opposite sides are equal: (I. 34.)  
 and because  $AD$  is equal to  $AB$ , (I. def. 30.)  
 and that  $AE$  is the half of  $AD$ , and  $AF$  the half of  $AB$ ,  
 therefore  $AE$  is equal to  $AF$ ; (ax. 7.)  
 wherefore the sides opposite to these are equal, viz.  $FG$  to  $GE$ :  
 in the same manner it may be demonstrated that  $GH, GK$  are each  
 of them equal to  $FG$  or  $GE$ :  
 therefore the four straight lines  $GE, GF, GH, GK$  are equal to one  
 another;  
 and the circle described from the center  $G$  at the distance of one of  
 them, will pass through the extremities of the other three, and touch  
 the straight lines  $AB, BC, CD, DA$ ;  
 because the angles at the points  $E, F, H, K$ , are right angles, (I. 29.)  
 and that the straight line which is drawn from the extremity of a  
 diameter, at right angles to it, touches the circle: (III. 16. Cor.)  
 therefore each of the straight lines  $AB, BC, CD, DA$  touches the circle,  
 which therefore is inscribed in the square  $ABCD$ . Q.E.F.

## PROPOSITION IX. PROBLEM.

To describe a circle about a given square.

Let  $ABCD$  be the given square.

It is required to describe a circle about  $ABCD$ .



Join  $AC$ ,  $BD$ , cutting one another in  $E$ :

and because  $DA$  is equal to  $AB$ , and  $AC$  common to the triangles  $DAC$ ,  $BAC$ , (I. def. 30.)

the two sides  $DA$ ,  $AC$  are equal to the two  $BA$ ,  $AC$ , each to each;

and the base  $DC$  is equal to the base  $BC$ ;

wherefore the angle  $DAC$  is equal to the angle  $BAC$ ; (I. 8.)

and the angle  $DAB$  is bisected by the straight line  $AC$ :

in the same manner it may be demonstrated that the angles  $ABC$ ,

$BCD$ ,  $CDA$  are severally bisected by the straight lines  $BD$ ,  $AC$ :

therefore, because the angle  $DAB$  is equal to the angle  $ABC$ , (I. def. 30.)

and that the angle  $EAB$  is the half of  $DAB$ , and  $EBA$  the half of  $ABC$ ;

therefore the angle  $EAB$  is equal to the angle  $EBA$ ; (ax. 7.)

wherefore the side  $EA$  is equal to the side  $EB$ : (I. 6.)

in the same manner it may be demonstrated, that the straight lines

$EC$ ,  $ED$  are each of them equal to  $EA$  or  $EB$ :

therefore the four straight lines  $EA$ ,  $EB$ ,  $EC$ ,  $ED$  are equal to one another;

and the circle described from the center  $E$ , at the distance of one of them, will pass through the extremities of the other three, and be described about the square  $ABCD$ . Q. E. F.

## PROPOSITION X. PROBLEM.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Take any straight line  $AB$ , and divide it in the point  $C$ , (II. 11.)

so that the rectangle  $AB$ ,  $BC$  may be equal to the square on  $CA$ ;

and from the center  $A$ , at the distance  $AB$ , describe the circle  $BDE$ , in which place the straight line  $BD$  equal to  $AC$ , which is not greater than the diameter of the circle  $BDE$ ; (IV. 1.)

and join  $DA$ .

Then the triangle  $ABD$  shall be such as is required, that is, each of the angles  $ABD$ ,  $ADB$  shall be double of the angle  $BAD$ .

Join  $DC$ , and about the triangle  $ADC$  describe the circle  $ACD$ . (IV. 5.)

And because the rectangle  $AB$ ,  $BC$  is equal to the square on  $AC$ ,

and that  $AC$  is equal to  $BD$ , (constr.)

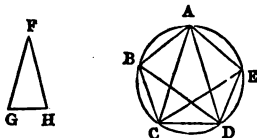
the rectangle  $AB$ ,  $BC$  is equal to the square on  $BD$ : (ax. 1.)

and because from the point  $B$ , without the circle  $ACD$ , two straight lines  $BCA$ ,  $BD$  are drawn to the circumference, one of which cuts, and





wherefore each of the angles  $ACD$ ,  $CDA$  is double of the angle  $CAD$ .  
 Bisect the angles  $ACD$ ,  $CDA$  by the straight lines  $CE$ ,  $DB$ ; (I. 9.)  
 and join  $AB$ ,  $BC$ ,  $DE$ ,  $EA$ .



Then  $ABCDE$  shall be the pentagon required.  
 Because each of the angles  $ACD$ ,  $CDA$  is double of  $CAD$ ,  
 and that they are bisected by the straight lines  $CE$ ,  $DB$ ;  
 therefore the five angles  $DAC$ ,  $ACE$ ,  $ECD$ ,  $CDB$ ,  $BDA$  are  
 equal to one another:  
 but equal angles stand upon equal circumferences; (III. 26.)  
 therefore the five circumferences  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$  are equal  
 to one another:  
 and equal circumferences are subtended by equal straight lines; (III. 29.)  
 therefore the five straight lines  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$  are equal  
 to one another.

Wherefore the pentagon  $ABCDE$  is equilateral.

It is also equiangular:

for, because the circumference  $AB$  is equal to the circumference  $DE$ ,  
 if to each be added  $BCD$ ,  
 the whole  $ABCD$  is equal to the whole  $EDCB$ : (ax. 2.)  
 but the angle  $AED$  stands on the circumference  $ABCD$ ;  
 and the angle  $BAE$  on the circumference  $EDCB$ ;  
 therefore the angle  $BAE$  is equal to the angle  $AED$ : (III. 27.)  
 for the same reason, each of the angles  $ABC$ ,  $BCD$ ,  $CDE$  is equal  
 to the angle  $BAE$ , or  $AED$ :  
 therefore the pentagon  $ABCDE$  is equiangular;  
 and it has been shewn that it is equilateral:  
 wherefore, in the given circle, an equilateral and equiangular pentagon  
 has been described. Q. E. F.

## PROPOSITION XII. PROBLEM.

*To describe an equilateral and equiangular pentagon about a given circle.*

Let  $ABCDE$  be the given circle.

It is required to describe an equilateral and equiangular pentagon about the circle  $ABCDE$ .

Let the angular points of a pentagon, inscribed in the circle, by the last proposition, be in the points  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  
 so that the circumferences  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$  are equal; (IV. 11.)  
 and through the points  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$  draw  $GH$ ,  $HK$ ,  $KL$ ,  $LM$ ,  
 $MG$  touching the circle; (III. 17.)

the figure  $GHKLM$  shall be the pentagon required.

Take the center  $F$ , and join  $FB$ ,  $FK$ ,  $FC$ ,  $FL$ ,  $FD$ .

And because the straight line  $KL$  touches the circle  $ABCDE$  in the point  $C$ , to which  $FC$  is drawn from the center  $F$ ,  
 $FC$  is perpendicular to  $KL$ , (III. 18.)

therefore each of the angles at  $C$  is a right angle:  
for the same reason, the angles at the points  $B, D$  are right angles:



and because  $FCK$  is a right angle,  
the square on  $FK$  is equal to the squares on  $FC, CK$ : (I. 47.)  
for the same reason, the square on  $FK$  is equal to the squares on  
 $FB, BK$ :  
therefore the squares on  $FC, CK$  are equal to the squares on  $FB, BK$ ; (ax. 1.)  
of which the square on  $FC$  is equal to the square on  $FB$ ;  
therefore the remaining square on  $CK$  is equal to the remaining square  
on  $BK$ , (ax. 3.) and the straight line  $CK$  equal to  $BK$ ;  
and because  $FB$  is equal to  $FC$ , and  $FK$  common to the triangles  
 $BFK, CFK$ ,  
the two  $BF, FK$  are equal to the two  $CF, FK$ , each to each:  
and the base  $BK$  was proved equal to the base  $KC$ :  
therefore the angle  $BFK$  is equal to the angle  $KFC$ , (I. 8.)  
and the angle  $BKF$  to  $FKC$ : (I. 4.)  
wherefore the angle  $BFC$  is double of the angle  $KFC$ ,  
and  $BKC$  double of  $FKC$ :  
for the same reason, the angle  $CFD$  is double of the angle  $CFL$ ,  
and  $CLD$  double of  $CLF$ :  
and because the circumference  $BC$  is equal to the circumference  $CD$ ,  
the angle  $BFC$  is equal to the angle  $CFD$ ; (III. 27.)  
and  $BFC$  is double of the angle  $KFC$ ,  
and  $CFD$  double of  $CFL$ ;  
therefore the angle  $KFC$  is equal to the angle  $CFL$ : (ax. 7.)  
and the right angle  $FCK$  is equal to the right angle  $FCL$ ;  
therefore, in the two triangles  $FKC, FLC$ , there are two angles of the  
one equal to two angles of the other, each to each;  
and the side  $FC$  which is adjacent to the equal angles in each, is com-  
mon to both;  
therefore the other sides are equal to the other sides, and the third  
angle to the third angle: (I. 26.)  
therefore the straight line  $KC$  is equal to  $CL$ , and the angle  $FKC$   
to the angle  $FLC$ :  
and because  $KC$  is equal to  $CL$ ,  
 $KL$  is double of  $KC$ .  
In the same manner it may be shewn that  $HK$  is double of  $BK$ :  
and because  $BK$  is equal to  $KC$ , as was demonstrated,  
and that  $KL$  is double of  $KC$ , and  $HK$  double of  $BK$ ,  
therefore  $HK$  is equal to  $KL$ : (ax. 6.)  
in like manner it may be shewn that  $GH, GM, ML$  are each of them  
equal to  $HK$ , or  $KL$ :

therefore the pentagon  $G HKLM$  is equilateral.

It is also equiangular :

for, since the angle  $FKC$  is equal to the angle  $FLC$ ,  
and that the angle  $HKL$  is double of the angle  $FKC$ ,  
and  $KLM$  double of  $FLC$ , as was before demonstrated;  
therefore the angle  $HKL$  is equal to  $KLM$ : (ax. 6.)

and in like manner it may be shewn,  
that each of the angles  $KHG$ ,  $HGM$ ,  $GML$  is equal to the angle  
 $HKL$  or  $KLM$ :

therefore the five angles  $G HK, HKL, KLM, LMG, MGH$  being  
equal to one another,

the pentagon  $G HKLM$  is equiangular :

and it is equilateral, as was demonstrated;

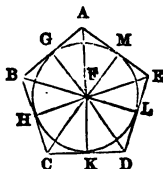
and it is described about the circle  $ABCDE$ . Q.E.F.

### PROPOSITION XIII. PROBLEM.

*To inscribe a circle in a given equilateral and equiangular pentagon.*

Let  $ABCDE$  be the given equilateral and equiangular pentagon.

It is required to inscribe a circle in the pentagon  $ABCDE$ .



Bisect the angles  $BCD, CDE$  by the straight lines  $CF, DF$ , (I. 9.)  
and from the point  $F$ , in which they meet, draw the straight lines  $FB$ ,  
 $FA, FE$ :

therefore since  $BC$  is equal to  $CD$ , (hyp.)

and  $CF$  common to the triangles  $BCF, DCF$ ,

the two sides  $BC, CF$  are equal to the two  $DC, CF$ , each to each;

and the angle  $BCF$  is equal to the angle  $DCF$ ; (constr.)

therefore the base  $BF$  is equal to the base  $FD$ , (I. 4.)

and the other angles to the other angles, to which the equal sides are  
opposite:

therefore the angle  $CBF$  is equal to the angle  $CDF$ :

and because the angle  $CDE$  is double of  $CDF$ ,

and that  $CDE$  is equal to  $CBA$ , and  $CDF$  to  $CBF$ ;

$CBA$  is also double of the angle  $CBF$ ;

therefore the angle  $ABF$  is equal to the angle  $CBF$ ;

wherefore the angle  $ABC$  is bisected by the straight line  $BF$ :

in the same manner it may be demonstrated,

that the angles  $BAE, AED$ , are bisected by the straight lines  $AF, FE$ .

From the point  $F$ , draw  $FG, FH, FK, FL, FM$  perpendiculars to  
the straight lines  $AB, BC, CD, DE, EA$ : (I. 12.)

and because the angle  $HCF$  is equal to  $KCF$ , and the right angle  
 $FHC$  equal to the right angle  $FKC$ ;

therefore in the triangles  $FHC$ ,  $FKC$ , there are two angles of the one equal to two angles of the other, each to each;  
and the side  $FC$ , which is opposite to one of the equal angles in each, is common to both;

therefore the other sides are equal, each to each; (I. 26.)

wherefore the perpendicular  $FH$  is equal to the perpendicular  $FK$ :  
in the same manner it may be demonstrated, that  $FL$ ,  $FM$ ,  $FG$  are each of them equal to  $FH$ , or  $FK$ :

therefore the five straight lines  $FG$ ,  $FH$ ,  $FK$ ,  $FL$ ,  $FM$  are equal to one another:

wherefore the circle described from the center  $F$ , at the distance of one of these five, will pass through the extremities of the other four, and touch the straight lines  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$ ,

because the angles at the points  $G$ ,  $H$ ,  $K$ ,  $L$ ,  $M$  are right angles, and that a straight line drawn from the extremity of the diameter of a circle at right angles to it, touches the circle; (III. 16.)

therefore each of the straight lines  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$  touches the circle:

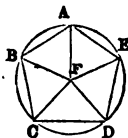
wherefore it is inscribed in the pentagon  $ABCDE$ . Q. E. F.

#### PROPOSITION XIV. PROBLEM.

*To describe a circle about a given equilateral and equiangular pentagon.*

Let  $ABCDE$  be the given equilateral and equiangular pentagon.

It is required to describe a circle about  $ABCDE$ .



Bisect the angles  $BCD$ ,  $CDE$  by the straight lines  $CF$ ,  $FD$ , (I. 9.)  
and from the point  $F$ , in which they meet, draw the straight lines  $FB$ ,  $FA$ ,  $FE$ , to the points  $B$ ,  $A$ ,  $E$ .

It may be demonstrated, in the same manner as the preceding proposition,

that the angles  $CBA$ ,  $BAE$ ,  $AED$  are bisected by the straight lines  $FB$ ,  $FA$ ,  $FE$ .

And because the angle  $BCD$  is equal to the angle  $CDE$ ,

and that  $FC$  is the half of the angle  $BCD$ ,

and  $CD$  the half of  $CDE$ ;

therefore the angle  $FCD$  is equal to  $FDC$ ; (ax. 7.)

wherefore the side  $CF$  is equal to the side  $FD$ : (I. 6.)

in like manner it may be demonstrated,

that  $FB$ ,  $FA$ ,  $FE$ , are each of them equal to  $FC$  or  $FD$ :

therefore the five straight lines  $FA$ ,  $FB$ ,  $FC$ ,  $FD$ ,  $FE$ , are equal to one another;

and the circle described from the center  $F$ , at the distance of one of them, will pass through the extremities of the other four, and be described about the equilateral and equiangular pentagon  $ABCDE$ .

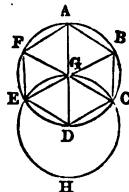
Q. E. F.

## PROPOSITION XV. PROBLEM.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let  $ABCDEF$  be the given circle.

It is required to inscribe an equilateral and equiangular hexagon in it.



Find the center  $G$  of the circle  $ABCDEF$ ,  
and draw the diameter  $AGD$ ; (III. 1.)  
and from  $D$ , as a center, at the distance  $DG$ , describe the circle  $EGCH$ ,  
join  $EG$ ,  $CG$ , and produce them to the points  $B$ ,  $F$ ;  
and join  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EF$ ,  $FA$ :  
the hexagon  $ABCDEF$  shall be equilateral and equiangular.

Because  $G$  is the center of the circle  $ABCDEF$ ,

$GE$  is equal to  $GD$ :

and because  $D$  is the center of the circle  $EGCH$ ,

$DE$  is equal to  $DG$ :

wherefore  $GE$  is equal to  $ED$ , (ax. 1.)

and the triangle  $EGD$  is equilateral;

and therefore its three angles  $EGD$ ,  $GDE$ ,  $DEG$ , are equal to one another: (I. 5. Cor.)

but the three angles of a triangle are equal to two right angles; (I. 32.)

therefore the angle  $EGD$  is the third part of two right angles:

in the same manner it may be demonstrated,

that the angle  $DGC$  is also the third part of two right angles:

and because the straight line  $GC$  makes with  $EB$  the adjacent angles  $EGC$ ,  $CGB$  equal to two right angles; (I. 13.)

the remaining angle  $CGB$  is the third part of two right angles:

therefore the angles  $EGD$ ,  $DGC$ ,  $CGB$  are equal to one another:

and to these are equal the vertical opposite angles  $BGA$ ,  $AGF$ ,  $FGE$ :  
(I. 15.)

therefore the six angles  $EGD$ ,  $DGC$ ,  $CGB$ ,  $BGA$ ,  $AGF$ ,  $FGE$ ,  
are equal to one another:

but equal angles stand upon equal circumferences; (III. 26.)

therefore the six circumferences  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EF$ ,  $FA$  are equal  
to one another:

and equal circumferences are subtended by equal straight lines:  
(III. 29.)

therefore the six straight lines are equal to one another,

and the hexagon  $ABCDEF$  is equilateral.

It is also equiangular:

for, since the circumference  $AF$  is equal to  $ED$ ,

to each of these equals add the circumference  $ABCD$ ;

therefore the whole circumference  $FABCD$  is equal to the whole  
 $EDCBA$ :

and the angle  $FED$  stands upon the circumference  $FABCD$ ,

and the angle  $AFE$  upon  $EDCBA$ ;

therefore the angle  $AFE$  is equal to  $FED$ : (III. 27.)

in the same manner it may be demonstrated,

that the other angles of the hexagon  $ABCDEF$  are each of them equal to the angle  $AFE$  or  $FED$ : therefore the hexagon is equiangular; and it is equilateral, as was shewn;

and it is inscribed in the given circle  $ABCDEF$ . Q.E.F.

COR.—From this it is manifest, that the side of the hexagon is equal to the straight line from the center, that is, to the semi-diameter of the circle.

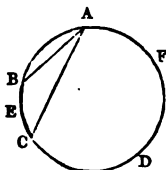
And if through the points  $A, B, C, D, E, F$  there be drawn straight lines touching the circle, an equilateral and equiangular hexagon will be described about it, which may be demonstrated from what has been said of the pentagon: and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like to that used for the pentagon.

#### PROPOSITION XVI. PROBLEM.

*To inscribe an equilateral and equiangular quindecagon in a given circle.*

Let  $ABCD$  be the given circle.

It is required to inscribe an equilateral and equiangular quindecagon in the circle  $ABCD$ .



Let  $AC$  be the side of an equilateral triangle inscribed in the circle, (IV. 2.) and  $AB$  the side of an equilateral and equiangular pentagon inscribed in the same; (IV. 11.)

therefore, of such equal parts as the whole circumference  $ABCDF$  contains fifteen,

the circumference  $ABC$ , being the third part of the whole, contains five;

and the circumference  $AB$ , which is the fifth part of the whole, contains three;

therefore  $BC$ , their difference, contains two of the same parts:

bisect  $BC$  in  $E$ ; (III. 30.)

therefore  $BE, EC$  are, each of them, the fifteenth part of the whole circumference  $ABCD$ :

therefore if the straight lines  $BE, EC$  be drawn, and straight lines equal to them be placed round in the whole circle, (IV. 1.) an equilateral and equiangular quindecagon will be inscribed in it. Q.E.F.

And in the same manner as was done in the pentagon, if through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon will be described about it: and likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

## NOTES TO BOOK IV.

THE Fourth Book of the Elements contains some particular cases of four general problems on the inscription and the circumscription of triangles and regular figures in and about circles. Euclid has not given any instance of the inscription or circumscription of rectilinear figures in and about other rectilinear figures.

Any rectilinear figure, of five sides and angles, is called a pentagon; of seven sides and angles, a heptagon; of eight sides and angles, an octagon; of nine sides and angles, a nonagon; of ten sides and angles, a decagon; of eleven sides and angles, an undecagon; of twelve sides and angles, a duodecagon; of fifteen sides and angles, a quindecagon, &c.

These figures are included under the general name of *polygons*; and are called *equilateral*, when their sides are equal; and *equiangular*, when their angles are equal; also when both their sides and angles are equal, they are called *regular polygons*.

Prop. III. An objection has been raised to the construction of this problem. It is said that in this and other instances of a similar kind, the lines which touch the circle at  $A$ ,  $B$ , and  $C$ , should be proved to meet one another. This may be done by joining  $AB$ , and then since the angles  $KAM$ ,  $KBM$  are equal to two right angles (III. 18.), therefore the angles  $BAM$ ,  $ABM$  are less than two right angles, and consequently (ax. 12.),  $AM$  and  $BM$  must meet one another, when produced far enough. Similarly, it may be shewn that  $AL$  and  $CL$ , as also  $CN$  and  $BN$  meet one another.

Prop. v. is the same as "To describe a circle passing through three given points, provided that they are not in the same straight line."

The corollary to this proposition appears to have been already demonstrated in Prop. 31. Book III.

It is obvious that the square described about a circle is equal to double the square inscribed in the same circle. Also that the circumscribed square is equal to the square on the diameter, or four times the square on the radius of the circle.

Prop. VII. It is manifest that a square is the only right-angled parallelogram which can be circumscribed about a circle, but that both a rectangle and a square may be inscribed in a circle.

Prop. x. By means of this proposition, a right angle may be divided into five equal parts.

Reference has already been made to the distinction between *analysis* and *synthesis*, and that all Euclid's *direct* demonstrations are *synthetic*, properly so called. There is however a single exception in Prop. 16. Book IV, where the analysis only is given of the Problem. The two methods are so connected in all processes of reasoning, that it is very difficult to separate one from the other, and to assert that *this process* is really *synthetic*, and *that* is really *analytic*. In every operation performed in the construction of a problem, there must be in the mind a knowledge of some properties of the figure which suggest the steps to be taken in the construction of it. Let any Problem be selected from Euclid, and at each step of the operation, let the question be asked, "Why that step is taken?" It will be found that it is *because* of some known property of the required figure. As an example will make the subject more clear to the learner, the Analysis of Euc. IV. 10, is taken from the "Analysis of Problems" in the larger edition of the Euclid, and to which the learner is referred for more complete information.

In Euc. IV. 10, there are five operations specified in the construction:—

- (1) Take any straight line  $AB$ .

(2) Divide the line  $AB$  in  $C$ , so that the rectangle  $AB, BC$ , may be equal to the square on  $AC$ .

(3) Describe the circle  $BDE$  with center  $A$  and radius  $AB$ .

(4) Place the line  $BD$  in that circle, equal to the line  $AC$ .

(5) Join the points  $A, D$ .

Why should either of these operations be performed rather than any others? And what will enable us to foresee that the result of them will be such a triangle as was required? The demonstration affixed to it by Euclid does undoubtedly prove that these operations must, in conjunction, produce such a triangle: but we are furnished in the Elements with no obvious reason for the adoption of these steps, unless we suppose them accidental. To suppose that all the constructions, even the simpler ones, are the result of accident only, would be supposing more than could be shewn to be admissible. No construction of the problem could have been devised without a previous knowledge of some of the properties of the figure. In fact, in directing the figure to be constructed, we assume the possibility of its existence; and we study the properties of such a figure on the hypothesis of its actual existence. It is this study of the properties of the figure that constitutes the *Analysis of the problem*.

Let then the existence of a triangle  $BAD$  be admitted, which has each of the angles  $ABD, ADB$  double of the angle  $BAD$ , in order to ascertain any properties it may possess which would assist in the construction of such a triangle.

Then, since the angle  $ADB$  is double of  $BAD$ , if we draw a line  $DC$  to bisect  $ADB$  and meet  $AB$  in  $C$ , the angle  $ADC$  will be equal to  $CAD$ ; and hence (Euc. I. 6.) the sides  $AC, CD$  are equal to one another.

Again, since we have three points  $A, C, D$ , not in the same straight line, let us examine the effect of describing a circle through them: that is, describe the circle  $ACD$  about the triangle  $ACD$ . (Euc. IV. 6.)

Then, since the angle  $ADB$  has been bisected by  $DC$ , and since  $ADB$  is double of  $DAB$ , the angle  $CDB$  is equal to the angle  $DAC$  in the alternate segment of the circle; the line  $BD$  therefore coincides with a tangent to the circle at  $D$ . (Converse of Euc. III. 32.)

Whence it follows, that the rectangle contained by  $AB, BC$ , is equal to the square on  $BD$ . (Euc. III. 36.)

But the angle  $BCD$  is equal to the two interior opposite angles  $CAD, CDA$ ; or since these are equal to each another,  $BCD$  is the double of  $CAD$ , that is, of  $BAD$ . And since  $ABD$  is also double of  $BAD$ , by the conditions of the triangle, the angles  $BCD, CBD$  are equal, and  $BD$  is equal to  $DC$ , that is, to  $AC$ .

It has been proved that the rectangle  $AB, BC$ , is equal to the square on  $BD$ ; and hence the point  $C$  in  $AB$ , found by the intersection of the bisecting line  $DC$ , is such, that the rectangle  $AB, BC$  is equal to the square on  $AC$ . (Euc. II. 11.)

Finally, since the triangle  $ABD$  is isosceles, having each of the angles  $ABD, ADB$  double of the same angle, the sides  $AB, AD$  are equal, and hence the points  $B, D$ , are in the circumference of the circle described about  $A$  with the radius  $AB$ . And since the magnitude of the triangle is not specified, the line  $AB$  may be of any length whatever.

From this "Analysis of the problem," which obviously is nothing more than an examination of the properties of such a figure supposed to exist already, it will be at once apparent, *why* those steps which are prescribed by Euclid for its construction, were adopted.

The line  $AB$  is taken of any length, *because* the problem does not prescribe any specific magnitude to any of the sides of the triangle.



The circle  $BDE$  is described about  $A$  with the distance  $AB$ , *because* the triangle is to be isosceles, having  $AB$  for one side, and therefore the other extremity of the base is in the circumference of that circle.

The line  $AB$  is divided in  $C$ , so that the rectangle  $AB, BC$  shall be equal to the square on  $AC$ , *because* the base of the triangle must be equal to the segment  $AC$ .

And the line  $AD$  is drawn, *because* it completes the triangle, two of whose sides,  $AB, BD$  are already drawn.

Whenever we have reduced the construction to depend upon problems which have been already constructed, our analysis may be terminated; as was the case where, in the preceding example, we arrived at the division of the line  $AB$  in  $C$ ; this problem having been already constructed as the eleventh of the second book.

Prop. xvi. The arc subtending a side of the quindecagon, may be found by placing in the circle from the same point, two lines respectively equal to the sides of the regular hexagon and pentagon.

The centers of the inscribed and circumscribed circles of any regular polygon are coincident.

Besides the circumscription and inscription of triangles and regular polygons about and in circles, some very important problems are solved in the constructions respecting the division of the circumferences of circles into equal parts.

By inscribing an equilateral triangle, a square, a pentagon, a hexagon, &c. in a circle, the circumference is divided into three, four, five, six, &c. equal parts. In Prop. 26, Book III, it has been shewn that equal angles at the centers of equal circles, and therefore at the center of the same circle, subtend equal arcs; by bisecting the angles at the center, the arcs which are subtended by them are also bisected, and hence, a sixth, eighth, tenth, twelfth, &c. part of the circumference of a circle may be found.

If the right angle be considered as divided into 90 degrees, each degree into 60 minutes, and each minute into 60 seconds, and so on, according to the sexagesimal division of a degree; by the aid of the first corollary to Prop. 32, Book I, may be found the numerical magnitude of an interior angle of any regular polygon whatever.

Let  $\theta$  denote the magnitude of one of the interior angles of a regular polygon of  $n$  sides,

then  $n\theta$  is the sum of all the interior angles.

But all the interior angles of any rectilinear figure together with four right angles, are equal to twice as many right angles as the figure has sides, that is, if  $\pi$  be assumed to designate two right angles,

$$\therefore n\theta + 2\pi = n\pi,$$

$$\text{and } n\theta = n\pi - 2\pi = (n - 2) \cdot \pi,$$

$$\therefore \theta = \frac{(n - 2)}{n} \cdot \pi,$$

the magnitude of an interior angle of a regular polygon of  $n$  sides.

By taking  $n = 3, 4, 5, 6$ , &c. may be found the magnitude in terms of two right angles, of an interior angle of any regular polygon whatever.

Pythagoras was the first, as Proclus informs us in his commentary, who discovered that a multiple of the angles of three regular figures only, namely, the trigon, the square, and the hexagon, can fill up space round a point in a plane.

It has been shewn that the interior angle of any regular polygon of  $n$

sides in terms of two right angles, is expressed by the equation

$$\theta = \frac{n-2}{n} \cdot \pi.$$

Let  $\theta_3$  denote the magnitude of the interior angle of a regular figure of three sides, in which case,  $n = 3$ .

$$\text{Then } \theta_3 = \frac{3-2}{3} \cdot \pi = \frac{\pi}{3} = \text{one third of two right angles,}$$

$$\therefore 3\theta_3 = \pi,$$

$$\text{and } 6\theta_3 = 2\pi,$$

that is, six angles, each equal to the interior angle of an equilateral triangle, are equal to four right angles, and therefore six equilateral triangles may be placed so as completely to fill up the space round the point at which they meet in a plane.

In a similar way, it may be shewn that four squares and three hexagons may be placed so as completely to fill up the space round a point.

Also it will appear from the results deduced, that no other regular figures besides these three, can be made to fill up the space round a point; for any multiple of the interior angles of any other regular polygon, will be found to be in excess above, or in defect from four right angles.

The equilateral triangle or trigon, the square or tetragon, the pentagon, and the hexagon, were the only regular polygons known to the Greeks, capable of being inscribed in circles, besides those which may be derived from them.

M. Gauss in his *Disquisitiones Arithmeticae*, has extended the number by shewing that in general, a regular polygon of  $2^n + 1$  sides is capable of being inscribed in a circle by means of straight lines and circles, in those cases in which  $2^n + 1$  is a prime number.

The case in which  $n = 4$ , in  $2^n + 1$ , was proposed by Mr. Lowry of the Royal Military College, to be answered in the seventeenth number of *Leybourn's Mathematical Repository*, in the following form:—

Required a geometrical demonstration of the following method of constructing a regular polygon of seventeen sides in a circle.

Draw the radius  $CO$  at right angles to the diameter  $AB$ ; on  $OC$  and  $OB$ , take  $OQ$  equal to the half, and  $OD$  equal to the eighth part of the radius; make  $DE$  and  $DF$  each equal to  $DQ$ , and  $EG$  and  $FH$  respectively equal to  $EQ$  and  $FQ$ ; take  $OK$  a mean proportional between  $OH$  and  $OQ$ , and through  $K$ , draw  $KM$  parallel to  $AB$ , meeting the semicircle described on  $OG$  in  $M$ , draw  $MN$  parallel to  $OC$  cutting the given circle in  $N$ , the arc  $AN$  is the seventeenth part of the whole circumference.

A demonstration of the truth of this construction has been given by Mr. Lowry himself, and will be found in the fourth volume of *Leybourn's Repository*. The demonstration including the two lemmas occupies more than eight pages, and is by no means of an elementary character.

## QUESTIONS ON BOOK IV.

1. WHAT is the general object of the Fourth Book of Euclid?
2. What consideration renders necessary the first proposition of the Fourth Book of Euclid?
3. When is a circle said to be inscribed within, and circumscribed about a rectilineal figure?

4. When is one rectilineal figure said to be inscribed in, and circumscribed about another rectilineal figure?

5. Modify the construction of *Euc. iv. 4*, so that the circle may touch one side of the triangle and the other two sides produced.

6. The sides of a triangle are 5, 6, 7 units respectively, find the radii of the inscribed and circumscribed circle.

7. Give the constructions by which the centers of circles described about, and inscribed in triangles are found. In what triangles will they coincide?

8. How is it shown that the radius of the circle inscribed in an equilateral triangle is half the radius described about the same triangle?

9. The equilateral triangle inscribed in a circle is one-fourth of the equilateral triangle circumscribed about the same circle.

10. What relation subsists between the square inscribed in, and the square circumscribed about the same circle?

11. Enunciate *Euc. iii. 22*: and extend this property to any inscribed polygon having an even number of sides.

12. Trisect a quadrantal arc of a circle, and show that every arc which is an  $\frac{m}{2^n}$ th part of a quadrantal arc may be trisected geometrically:  $m$  and  $n$  being whole numbers:

13. If one side of a quadrilateral figure inscribed in a circle be produced, the exterior angle is equal to the interior and opposite angle of the figure. Is this property true of any inscribed polygon having an even number of sides?

14. In what parallelograms can circles be inscribed?

15. Give the analysis and synthesis of the problem: to describe an isosceles triangle, having each of the angles at the base double of the third angle?

16. Shew that in the figure *Euc. iv. 10*, there are two triangles possessing the required property.

17. How is it made to appear that the line *BD* is the side of a regular decagon inscribed in the larger circle, and the side of a regular pentagon inscribed in the smaller circle? *fig. Euc. iv. 10*.

18. In the construction of *Euc. iv. 3*, Euclid has omitted to shew that the tangents drawn through the points *A* and *B* will meet in some point *M*. How may this be shewn?

19. Shew that if the points of intersection of the circles in Euclid's figure, *Book iv. Prop. 10*, be joined with the vertex of the triangle and with each other, another triangle will be formed equiangular and equal to the former.

20. Divide a right angle into five equal parts. How may an isosceles triangle be described upon a given base, having each angle at the base one-third of the angle at the vertex?

21. What regular figures may be inscribed in a circle by the help of *Euc. iv. 10*?

22. What is Euclid's definition of a regular pentagon? Would the stellated figure, which is formed by joining the alternate angles of a regular pentagon, as described in the Fourth Book, satisfy this definition?

23. Shew that each of the interior angles of a regular pentagon inscribed in a circle, is equal to three-fifths of two right angles.

24. If two sides not adjacent, of a regular pentagon, be produced to meet: what is the magnitude of the angle contained at the point where they meet?

25. Is there any method more direct than Euclid's for inscribing a regular pentagon in a circle?

26. In what sense is a regular hexagon also a parallelogram? Would the same observation apply to all regular figures with an even number of sides?

27. Why has Euclid not shewn how to inscribe an equilateral triangle in a circle, before he requires the use of it in Prop. 16, Book iv.?

28. An equilateral triangle is inscribed in a circle by joining the first, third, and fifth angles of the inscribed hexagon.

29. If the sides of a hexagon be produced to meet, the angles formed by these lines will be equal to four right angles.

30. Shew that the area of an equilateral triangle inscribed in a circle is one-half of a regular hexagon inscribed in the same circle.

31. If a side of an equilateral triangle be six inches: what is the radius of the inscribed circle?

32. Find the area of a regular hexagon inscribed in a circle whose diameter is twelve inches. What is the difference between the inscribed and the circumscribed hexagon?

33. Which is the greater, the difference between the side of the square and the side of the regular hexagon inscribed in a circle whose radius is unity; or the difference between the side of the equilateral triangle and the side of the regular pentagon inscribed in the same circle?

34. The regular hexagon inscribed in a circle, is three-fourths of the regular circumscribed hexagon.

35. Are the interior angles of an octagon equal to twelve right angles?

36. What figure is formed by the production of the alternate sides of a regular octagon?

37. How many square inches are in the area of a regular octagon whose side is eight inches?

38. If an irregular octagon be capable of having a circle described about it, shew that the sums of the angles taken alternately are equal.

39. Find an algebraical formula for the number of degrees contained by an interior angle of a regular polygon of  $n$  sides.

40. What are the three regular figures which can be used in paving a plane area? Shew that no other regular figures but these will fill up the space round a point in a plane.

41. Into what number of equal parts may a right angle be divided geometrically? What connection has the solution of this problem with the possibility of inscribing regular figures in circles?

42. Assuming the demonstrations in Euc. iv, shew that any equilateral figure of  $3.2''$ ,  $4.2''$ ,  $5.2''$ , or  $15.2''$  sides may be inscribed in a circle, when  $n$  is any of the numbers, 0, 1, 2, 3, &c.

43. With a pair of compasses only, shew how to divide the circumference of a given circle into twenty-four equal parts.

44. Shew that if any polygon inscribed in a circle be equilateral, it must also be equiangular. Is the converse true?

45. Shew that if the circumference of a circle pass through three angular points of a regular polygon, it will pass through all of them.

46. Similar polygons are always equiangular: is the converse of this proposition true?

47. What are the limits to the *Geometrical* inscription of regular figures in circles? What does *Geometrical* mean when used in this way?

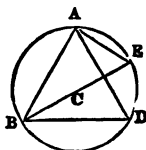
48. What is the difficulty of inscribing geometrically an equilateral and equiangular undecagon in a circle? Why is the solution of this problem said to be beyond the limits of plane geometry? Why is it so difficult to prove that the geometrical solution of such problems is impossible?

## GEOMETRICAL EXERCISES ON BOOK IV.

### PROPOSITION I. THEOREM.

*If an equilateral triangle be inscribed in a circle, the square on the side of the triangle is triple of the square on the radius, or on the side of the regular hexagon inscribed in the same circle.*

Let  $ABD$  be an equilateral triangle inscribed in the circle  $ABD$ , of which the center is  $C$ .



Join  $BC$ , and produce  $BC$  to meet the circumference in  $E$ , also join  $AE$ .

And because  $ABD$  is an equilateral triangle inscribed in the circle;  
therefore  $AED$  is one-third of the whole circumference,

and therefore  $AE$  is one-sixth of the circumference.

and consequently, the straight line  $AE$  is the side of a regular hexagon (IV. 15.), and is equal to  $EC$ .

And because  $BE$  is double of  $EC$  or  $AE$ .

therefore the square on  $BE$  is quadruple of the square on  $AE$ .

but the square on  $BE$  is equal to the squares on  $AB, AE$ ;

therefore the squares on  $AB$ ,  $AE$  are quadruple of the square on  $AE$ ,

and taking from these equals the square on  $AE$ ,

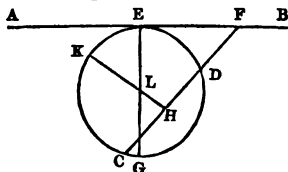
therefore the square on  $AB$  is triple of the square on  $AE$ .

**PROPOSITION II. PROBLEM.**

*To describe a circle which shall touch a straight line given in position, and pass through two given points.*

**Analysis.** Let  $AB$  be the given straight line, and  $C, D$  the two given points.

Suppose the circle required which passes through the points  $C, D$  to touch the line  $AB$  in the point  $E$ .



Join  $C, D$ , and produce  $DC$  to meet  $AB$  in  $F$ ,  
and let the circle be described having the center  $L$ ,

join also  $LE$ , and draw  $LH$  perpendicular to  $CD$ .

Then  $CD$  is bisected in  $H$ , and  $LE$  is perpendicular to  $AB$ .

Also, since from the point  $F$  without the circle, are drawn two straight lines, one of which  $FE$  touches the circle, and the other  $FDC$  cuts it; the rectangle contained by  $FC$ ,  $FD$ , is equal to the square on  $FE$ . (III. 36.)

Synthesis. Join  $C$ ,  $D$ , and produce  $CD$  to meet  $AB$  in  $F$ , take the point  $E$  in  $FB$ , such that the square on  $FE$ , shall be equal to the rectangle  $FD$ ,  $FC$ .

Bisect  $CD$  in  $H$ , and draw  $HK$  perpendicular to  $CD$ ;

then  $HK$  passes through the center. (III. 1, Cor. 1.)

At  $E$  draw  $EG$  perpendicular to  $FB$ ,

then  $EG$  passes through the center, (III. 19.)

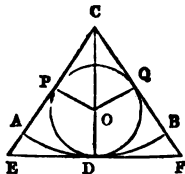
consequently  $L$ , the point of intersection of these two lines, is the center of the circle.

It is also manifest, that another circle may be described passing through  $C$ ,  $D$ , and touching the line  $AB$  on the other side of the point  $F$ ; and this circle will be equal to, greater than, or less than the other circle, according as the angle  $CFB$  is equal to, greater than, or less than the angle  $CFA$ .

### PROPOSITION III. PROBLEM.

*Inscribe a circle in a given sector of a circle.*

Analysis. Let  $CAB$  be the given sector, and let the required circle whose center is  $O$ , touch the radii in  $P$ ,  $Q$ , and the arc of the sector in  $D$ .



Join  $OP$ ,  $OQ$ , these lines are equal to one another.

Join also  $CO$ .

Then in the triangles  $CPO$ ,  $CQO$ , the two sides  $PC$ ,  $CO$ , are equal to  $QC$ ,  $CO$ , and the base  $OP$  is equal to the base  $OQ$ ;

therefore the angle  $PCO$  is equal to the angle  $QCO$ ;

and the angle  $ACB$  is bisected by  $CO$ ;

also  $CO$  produced will bisect the arc  $AB$  in  $D$ . (III. 26.)

If a tangent  $EDF$  be drawn to touch the arc  $AB$  in  $D$ ;

and  $CA$ ,  $CB$  be produced to meet it in  $E$ ,  $F$ ;

the inscription of the circle in the sector is reduced to the inscription of a circle in a triangle. (IV. 4.)

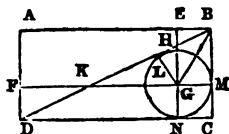
### PROPOSITION IV. PROBLEM.

$ABCD$  is a rectangular parallelogram. Required to draw  $EG$ ,  $FG$  parallel to  $AD$ ,  $DC$ , so that the rectangle  $EF$  may be equal to the figure  $EMD$ , and  $EB$  equal to  $FD$ .

Analysis. Let  $EG$ ,  $FG$  be drawn, as required, bisecting the rectangle  $ABCD$ .

Draw the diagonal  $BD$  cutting  $EG$  in  $H$  and  $FG$  in  $K$ .

Then  $BD$  also bisects the rectangle  $ABCD$ ;  
and therefore the area of the triangle  $KGH$  is equal to that of the two triangles  $EHB$ ,  $FKD$ .



Draw  $GL$  perpendicular to  $BD$ , and join  $GB$ ,  
also produce  $FG$  to  $M$ , and  $EG$  to  $N$ .

If the triangle  $LGH$  be supposed to be equal to the triangle  $EHB$ ,  
by adding  $HGB$  to each,

the triangles  $LGB$ ,  $GEB$  are equal, and they are upon the same  
base  $GB$ , and on the same side of it;

therefore they are between the same parallels,

that is, if  $L$ ,  $E$  were joined,  $LE$  would be parallel to  $GB$ ;

and if a semicircle were described on  $GB$  as a diameter, it would  
pass through the points  $E$ ,  $L$ ; for the angles at  $E$ ,  $L$  are right  
angles:

also  $LE$  would be a chord parallel to the diameter  $GB$ ;

therefore the arcs intercepted between the parallels  $LE$ ,  $GB$  are  
equal,

and consequently the chords  $EB$ ,  $LG$  are also equal;

but  $EB$  is equal to  $GM$ , and  $GM$  to  $GN$ ;

wherefore  $LG$ ,  $GM$ ,  $GN$ , are equal to one another;

hence  $G$  is the center of the circle inscribed in the triangle  $BDC$ .

Synthesis. Draw the diagonal  $BD$ .

Find  $G$  the center of the circle inscribed in the triangle  $BDC$ ;

through  $G$  draw  $EGN$  parallel to  $BC$ , and  $FKM$  parallel to  $AB$ .

Then  $EG$  and  $FG$  bisect the rectangle  $ABCD$ .

Draw  $GL$  perpendicular to the diagonal  $BD$ .

In the triangles  $GLH$ ,  $EHB$ , the angles  $GLH$ ,  $HEB$  are equal,  
each being a right angle, and the vertical angles  $LHG$ ,  $EHB$ , also the  
side  $LG$  is equal to the side  $EB$ ;

therefore the triangle  $LHG$  is equal to the triangle  $EHB$ .

Similarly, it may be proved, that the triangle  $GLK$  is equal to the  
triangle  $FKD$ ,

therefore the whole triangle  $KGH$  is equal to the two triangles  
 $EHB$ ,  $FKD$ ;

and consequently  $EG$ ,  $FG$  bisect the rectangle  $ABCD$ .

## I.

1. In a given circle, place a straight line equal and parallel to a given straight line not greater than the diameter of the circle.

2. Trisect a given circle by dividing it into three equal sectors.

3. The centers of the circle inscribed in, and circumscribed about an equilateral triangle coincide; and the diameter of one is twice the diameter of the other.

4. If a line be drawn from the vertex of an equilateral triangle, perpendicular to the base, and intersecting a line drawn from either of the angles at the base perpendicular to the opposite side; the distance from the vertex to the point of intersection, shall be equal to the radius of the circumscribing circle.

5. If an equilateral triangle be inscribed in a circle, and a straight line be drawn from the vertical angle to meet the circumference, it will be equal to the sum or difference of the straight lines drawn from the extremities of the base to the point where the line meets the circumference, according as the line does or does not cut the base.

6. The perpendicular from the vertex on the base of an equilateral triangle, is equal to the side of an equilateral triangle inscribed in a circle whose diameter is the base. Required proof.

7. If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected, the line joining the points of bisection shall be trisected by the sides.

8. If an equilateral triangle be inscribed in a circle, any of its sides will cut off one-fourth part of the diameter drawn through the opposite angle.

9. The perimeter of an equilateral triangle inscribed in a circle is greater than the perimeter of any other isosceles triangle inscribed in the same circle.

10. If any two consecutive sides of a hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other.

11. Prove that the area of a regular hexagon is greater than that of an equilateral triangle of the same perimeter.

12. If two equilateral triangles be inscribed in a circle so as to have the sides of one parallel to the sides of the other, the figure common to both will be a regular hexagon, whose area and perimeter will be equal to the remainder of the area and perimeter of the two triangles.

13. Determine the distance between the opposite sides of an equilateral and equiangular hexagon inscribed in a circle.

14. Inscribe a regular hexagon in a given equilateral triangle.

15. To inscribe a regular duodecagon in a given circle, and shew that its area is equal to the square on the side of an equilateral triangle inscribed in the circle.

## II.

16. Describe a circle touching three straight lines.

17. Any number of triangles having the same base and the same vertical angle, will be circumscribed by one circle.

18. Find a point in a triangle from which two straight lines



drawn to the extremities of the base shall contain an angle equal to twice the vertical angle of the triangle. Within what limitations is this possible?

19. Given the base of a triangle, and the point from which the perpendiculars on its three sides are equal; construct the triangle. To what limitation is the position of this point subject in order that the triangle may lie on the same side of the base?

20. From any point  $B$  in the radius  $CA$  of a given circle whose center is  $C$ , a straight line is drawn at right angles to  $CA$  meeting the circumference in  $D$ ; the circle described round the triangle  $CBD$  touches the given circle in  $D$ .

21. If a circle be described about a triangle  $ABC$ , and perpendiculars be let fall from the angular points  $A, B, C$ , on the opposite sides, and produced to meet the circle in  $D, E, F$ , respectively, the circumferences  $EF, FD, DE$ , are bisected in the points  $A, B, C$ .

22. If from the angles of a triangle, lines be drawn to the points where the inscribed circle touches the sides; these lines shall intersect in the same point.

23. The straight line which bisects any angle of a triangle inscribed in a circle, cuts the circumference in a point which is equidistant from the extremities of the side opposite to the bisected angle, and from the center of a circle inscribed in the triangle.

24. Let three perpendiculars from the angles of a triangle  $ABC$  on the opposite sides meet in  $P$ , a circle described so as to pass through  $P$  and any two of the points  $A, B, C$ , is equal to the circumscribing circle of the triangle.

25. If perpendiculars  $Aa, Bb, Cc$  be drawn from the angular points of a triangle  $ABC$  upon the opposite sides, shew that they will bisect the angles of the triangle  $abc$ , and thence prove that the perimeter of  $abc$  will be less than that of any other triangle which can be inscribed in  $ABC$ .

26. Find the least triangle which can be circumscribed about a given circle.

27. If  $ABC$  be a plane triangle,  $GCF$  its circumscribing circle, and  $GEF$  a diameter perpendicular to the base  $AB$ , then if  $CF$  be joined, the angle  $GFC$  is equal to half the difference of the angles at the base of the triangle.

28. The line joining the centers of the inscribed and circumscribed circles of a triangle, subtends at any one of the angular points an angle equal to the semi-difference of the other two angles.

### III.

29. The locus of the centers of the circles, which are inscribed in all right-angled triangles on the same hypotenuse, is the quadrant described on the hypotenuse.

30. The center of the circle which touches the two semicircles described on the sides of a right-angled triangle is the middle point of the hypotenuse.

31. If a circle be inscribed in a right-angled triangle, the excess of the sides containing the right angle above the hypotenuse is equal to the diameter of the inscribed circle.

32. Having given the hypotenuse of a right-angled triangle, and the radius of the inscribed circle, to construct the triangle.

33.  $ABC$  is a triangle inscribed in a circle, the line joining the middle points of the arcs  $AB$ ,  $AC$ , will cut off equal portions of the two contiguous sides measured from the angle  $A$ .

## IV.

34. Having given the vertical angle of a triangle, and the radii of the inscribed and circumscribed circles, to construct the triangle.

35. Given the base and vertical angle of a triangle, and also the radius of the inscribed circle, required to construct it.

36. Given the three angles of a triangle, and the radius of the inscribed circle, to construct the triangle.

37. If the base and vertical angle of a plane triangle be given, prove that the locus of the centers of the inscribed circle is a circle, and find its position and magnitude.

## V.

38. In a given triangle inscribe a parallelogram which shall be equal to one-half the triangle. Is there any limit to the number of such parallelograms?

39. In a given triangle to inscribe a triangle, the sides of which shall be parallel to the sides of a given triangle.

40. If any number of parallelograms be inscribed in a given parallelogram, the diameters of all the figures shall cut one another in the same point.

41. A square is inscribed in another, the difference of the areas is twice the rectangle contained by the segments of the side which are made at the angular point of the inscribed square.

42. Inscribe an equilateral triangle in a square, (1) When the vertex of the triangle is in an angle of the square. (2) When the vertex of the triangle is in the point of bisection of a side of the square.

43. On a given straight line describe an equilateral and equiangular octagon.

## VI.

44. Inscribe a circle in a rhombus.

45. Having given the distances of the centers of two equal circles which cut one another, inscribe a square in the space included between the two circumferences.

46. The square inscribed in a circle is equal to half the square described about the same circle.

47. The square is greater than any oblong inscribed in the same circle.

48. A circle having a square inscribed in it being given, to find a circle in which a regular octagon of a perimeter equal to that of the square, may be inscribed.

49. Describe a circle about a figure formed by constructing an equilateral triangle upon the base of an isosceles triangle, the vertical angle of which is four times the angle at the base.

50. A regular octagon inscribed in a circle is equal to the rectangle

contained by the sides of the squares inscribed in, and circumscribed about the circle.

51. If in any circle the side of an inscribed hexagon be produced till it becomes equal to the side of an inscribed square, a tangent drawn from the extremity, without the circle, shall be equal to the side of an inscribed octagon.

### VII.

52. To describe a circle which shall touch a given circle in a given point; and also a given straight line.

53. Describe a circle touching a given straight line, and also two given circles.

54. Describe a circle which shall touch a given circle, and each of two given straight lines.

55. Two points are given, one in each of two given circles; describe a circle passing through both points and touching one of the circles.

56. Describe a circle touching a straight line in a given point, and also touching a given circle. When the line cuts the given circle, shew that your construction will enable you to obtain six circles touching the given circle and the given line, but not necessarily in the given point.

57. Describe a circle which shall touch two sides and pass through one angle of a given square.

58. If two circles touch each other externally, describe a circle which shall touch one of them in a given point, and also touch the other. In what case does this become impossible?

59. Describe three circles touching each other and having their centers at three given points. In how many different ways may this be done?

### VIII.

60. Let two straight lines be drawn from any point within a circle to the circumference: describe a circle, which shall touch them both, and the arc between them.

61. In a given triangle having inscribed a circle, inscribe another circle in the space thus intercepted at one of the angles.

62. Let  $AB, AC$ , be the bounding radii of a quadrant; complete the square  $ABDC$  and draw the diagonal  $AD$ ; then the part of the diagonal without the quadrant will be equal to the radius of a circle inscribed in the quadrant.

63. If on one of the bounding radii of a quadrant, a semicircle be described, and on the other, another semicircle be described, so as to touch the former and the quadrantal arc; find the center of the circle inscribed in the figure bounded by the three curves.

64. In a given segment of a circle inscribe an isosceles triangle, such that its vertex may be in the middle of the chord, and the base and perpendicular together equal to a given line.

65. Inscribe three circles in an isosceles triangle touching each other, and each of them touching two of the three sides of the triangle.

### IX.

66. In the fig. Prop. 10, Book IV, shew that the base  $BD$  is the

side of a regular decagon inscribed in the larger circle, and the side of a regular pentagon inscribed in the smaller circle.

67. In the fig. Prop. 10, Book IV, produce  $DC$  to meet the circle in  $F$ , and draw  $BF$ ; then the angle  $ABF$  shall be equal to three times the angle  $BFD$ .

68. If the alternate angles of a regular pentagon be joined, the figure formed by the intersection of the joining lines will itself be a regular pentagon.

69. If  $ABCDE$  be any pentagon inscribed in a circle, and  $AC$ ,  $BD$ ,  $CE$ ,  $DA$ ,  $EB$  be joined, then are the angles  $ABE$ ,  $BCA$ ,  $CDB$ ,  $DEC$ ,  $EAD$ , together equal to two right angles.

70. A watch-ribbon is folded up into a flat knot of five edges, shew that the sides of the knot form an equilateral pentagon.

71. If from the extremities of the side of a regular pentagon inscribed in a circle, straight lines be drawn to the middle of the arc subtended by the adjacent side, their difference is equal to the radius; the sum of their squares to three times the square of the radius; and the rectangle contained by them is equal to the square of the radius.

72. Inscribe a regular pentagon in a given square so that four angles of the pentagon may touch respectively the four sides of the square.

73. Inscribe a regular decagon in a given circle.

74. The square described upon the side of a regular pentagon in a circle, is equal to the square on the side of a regular hexagon, together with the square upon the side of a regular decagon in the same circle.

#### X.

75. In a given circle inscribe three equal circles touching each other and the given circle.

76. Shew that if two circles be inscribed in a third to touch one another, the tangents of the points of contact will all meet in the same point.

77. If there be three concentric circles, whose radii are 1, 2, 3; determine how many circles may be described round the interior one, having their centers in the circumference of the circle, whose radius is 2, and touching the interior and exterior circles, and each other.

78. Shew that nine equal circles may be placed in contact, so that a square whose side is three times the diameter of one of them will circumscribe them.

#### XI.

79. Produce the sides of a given heptagon both ways, till they meet, forming seven triangles; required the sum of their vertical angles.

80. To convert a given regular polygon into another which shall have the same perimeter, but double the number of sides.

81. In any polygon of an even number of sides, inscribed in a circle, the sum of the 1st, 3rd, 5th, &c. angles is equal to the sum of the 2nd, 4th, 6th, &c.

82. Of all polygons having equal perimeters, and the same number of sides, the equilateral polygon has the greatest area.

## GEOMETRICAL EXERCISES ON BOOK I.

### HINTS, &c.

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8. This is a particular case of Euc. i. 22. The triangle however may be described by means of Euc. i. 1. Let AB be the given base, produce AB both ways to meet the circles in D, E (fig. Euc. i. 1.); with center A, and radius AE, describe a circle, and with center B and radius BD, describe another circle cutting the former in G. Join GA, GB.

9. Apply Euc. i. 6, 8.

10. This is proved by Euc. i. 32, 13, 5.

11. Let fall also a perpendicular from the vertex on the base.

12. Apply Euc. i. 4.

13. Let CAB be the triangle (fig. Euc. i. 10.) CD the line bisecting the angle ACD and the base AB. Produce CD, and make DE equal to CD, and join AE. Then CB may be proved equal to AE, also AE to AC.

14. Let AB be the given line, and C, D the given points. From C draw CE perpendicular to AB, and produce it making EF equal to CE, join FD, and produce it to meet the given line in G, which will be the point required.

15. Make the construction as the enunciation directs, then by Euc. i. 4, BH is proved equal to CK: and by Euc. i. 13, 6, OB is shewn to be equal to OC.

16. This proposition requires for its proof the case of equal triangles omitted in Euclid:—namely, when two sides and one angle are given, but not the angle included by the given sides.

17. The angle BCD may be shewn to be equal to the sum of the angles ABC, ADC.

18. The angles ADE, AED may be each proved to be equal to the complements of the angles at the base of the triangle.

19. The angles CAB, CBA, being equal, the angles CAD, CBE are equal, Euc. i. 13. Then, by Euc. i. 4, CD is proved to be equal to CE. And by Euc. i. 5, 32, the angle at the vertex is shewn to be four times either of the angles at the base.

20. Let AB, CD be two straight lines intersecting each other in E, and let P be the given point, within the angle AED. Draw EF bisecting the angle AED, and through P draw PGH parallel to EF, and cutting ED, EB in G, H. Then EG is equal to EH. And by bisecting the angle DEB and drawing through P a line parallel to this line, another solution is obtained. It will be found that the two lines are at right angles to each other.

21. Let the two given straight lines meet in A, and let P be the given point. Let PQR be the line required, meeting the lines AQ, AR in Q and R, so that PQ is equal to QR. Through P draw PS parallel to AR and join RS. Then APSR is a parallelogram and AS, PR the diagonals. Hence the construction.

22. Let the two straight lines AB, AC meet in A. In AB take any point D, and from AC cut off AE equal to AD, and join DE. On DE, or DE produced, take DF equal to the given line, and through F draw FG parallel to AB meeting AC in G, and through G draw GH parallel to DE meeting AB in H. Then GH is the line required.

23. The two given points may be both on the same side, or one point may be on each side of the line. If the point required in the line be supposed to be found, and lines be drawn joining this point and the given points, an isosceles triangle is formed, and if a perpendicular be drawn on the base from the point in the line: the construction is obvious.

24. The problem is simply this—to find a point in one side of a triangle from which the perpendiculars drawn to the other two sides shall be equal. If all the positions of these lines be considered, it will readily be seen in what case the problem is impossible.

25. If the *isosceles triangle* be obtuse-angled, by Euc. I. 5, 32, the truth will be made evident. If the triangle be acute-angled, the enunciation of the proposition requires some modification.

26. Construct the figure and apply Euc. I. 5, 32, 15.

If the isosceles triangle have its vertical angle less than two-thirds of a right-angle, the line ED produced, meets AB produced towards the base, and then  $3 \cdot \angle AEF = 4 \text{ right angles} + \angle AFE$ . If the vertical angle be greater than two-thirds of a right angle, ED produced meets AB produced towards the vertex, then  $3 \cdot \angle AEF = 2 \text{ right angles} + \angle AFE$ .

27. Let ABC be an isosceles triangle, and from any point D in the base BC, and the extremity B, let three lines DE, DF, BG be drawn to the sides and making equal angles with the base. Produce ED and make DH equal to DF and join BH.

28. In the isosceles triangle ABC, let the line DFE which meets the side AC in D and AB produced in E, be bisected by the base in the point E. Then DC may be shewn to be equal to BE.

29. If two equal straight lines be drawn terminated by two lines which meet in a point, they will cut off triangles of equal area. Hence the two triangles have a common vertical angle and their areas and bases equal. By Euc. I. 32 it is shewn that the angle contained by the bisecting lines is equal to the exterior angle at the base.

30. There is an omission in this question. After the words "making equal angles with the sides," add, "and be equal to each other respectively." (1), (3) Apply Euc. I. 26, 4. (2) The equal lines which bisect the sides may be shewn to make equal angles with the sides.

31. At C make the angle BCD equal to the angle ACB, and produce AB to meet CD in D.

32. By bisecting the hypotenuse, and drawing a line from the vertex to the point of bisection, it may be shewn that this line forms with the shorter side and half the hypotenuse an isosceles triangle.

33. Let ABC be a triangle, having the right angle at A, and the angle at C greater than the angle at B, also let AD be perpendicular to the base, and AE be the line drawn to E the bisection of the base. Then AE may be proved equal to BE or EC independently of Euc. III. 31.

34. Produce EG, FG to meet the perpendiculars CE, BF, produced if necessary. The demonstration is obvious.

35. If the given triangle have both of the angles at the base, acute angles; the difference of the angles at the base is at once obvious from Euc. I. 32. If one of the angles at the base be obtuse, does the property hold good?

36. Let ABC be a triangle having the angle ACB double of the angle ABC, and let the perpendicular AD be drawn to the base BC. Take DE equal to DC and join AE. Then AE may be proved to be equal to EB.

If ACB be an obtuse angle, then AC is equal to the sum of the segments of the base, made by the perpendicular from the vertex A.

37. Let the sides AB, AC of any triangle ABC be produced, the ex-

terior angles bisected by two lines which meet in D, and let AD be joined, then AD bisects the angle BAC. For draw DE perpendicular on BC, also DF, DG perpendiculars on AB, AC produced, if necessary. Then DF may be proved equal to DG, and the squares on DF, DA are equal to the squares on FG, GA, of which the square on FD is equal to the square on DG; hence AF is equal to AG, and Euc. I. 8, the angle BAC is bisected by AD.

38. The line required will be found to be equal to half the sum of the two sides of the triangle.

39. Apply Euc. I. 1, 9.

40. The angle to be trisected is one-fourth of a right angle. If an equilateral triangle be described on one of the sides of a triangle which contains the given angle, and a line be drawn to bisect that angle of the equilateral triangle which is at the given angle, the angle contained between this line and the other side of the triangle will be one-twelfth of a right angle, or equal to one-third of the given angle.

It may be remarked, generally, that any angle which is the half, fourth, eighth, &c. part of a right angle, may be trisected by Plane Geometry.

41. Apply Euc. I. 20.

42. Let ABC, DBC be two equal triangles on the same base, of which ABC is isosceles, fig. Euc. I. 37. By producing AB and making AG equal to AB or AC, and joining GD, the perimeter of the triangle ABC may be shewn to be less than the perimeter of the triangle DBC.

43. Apply Euc. I. 20.

44. For the first case, see Theo. 32, p. 76: for the other two cases, apply Euc. I. 19.

45. This is obvious from Euc. I. 26.

46. By Euc. I. 29, 6, FC may be shewn equal to each of the lines EF, FG.

47. Join GA and AF, and prove GA and AF to be in the same straight line.

48. Let the straight line drawn through D parallel to BC meet the side AB in E, and AC in F. Then in the triangle EBD, EB is equal to ED, by Euc. I. 29, 6. Also, in the triangle EAD, the angle EAD may be shewn equal to the angle EDA, whence EA is equal to ED, and therefore AB is bisected in E. In a similar way it may be shewn, by bisecting the angle C, that AC is bisected in F. Or the bisection of AC in F may be proved when AB is shewn to be bisected in E.

49. The triangle formed will be found to have its sides respectively parallel to the sides of the original triangle.

50. If a line equal to the given line be drawn from the point where the two lines meet, and parallel to the other given line; a parallelogram may be formed, and the construction effected.

51. Let ABC be the triangle; AD perpendicular to BC, AE drawn to the bisection of BC, and AF bisecting the angle BAC. Produce AD and make DA' equal to AD: join FA', EA'.

52. If the point in the base be supposed to be determined, and lines drawn from it parallel to the sides, it will be found to be in the line which bisects the vertical angle of the triangle.

53. Let ABC be the triangle, at C draw CD perpendicular to CB and equal to the sum of the required lines, through D draw DE parallel to CB meeting AC in E, and draw EF parallel to DC, meeting BC in F. Then EF is equal to DC. Next produce CB, making CG equal to CE, and join EG cutting AB in H. From H draw HK perpendicular to EAC, and

HL perpendicular to BC. Then HK and HL together are equal to DC. The proof depends on Theorem 27, p. 75.

54. Let  $C'$  be the intersection of the circles on the other side of the base, and join  $AC'$ ,  $BC'$ . Then the angles  $CBA$ ,  $C'BA$  being equal, the angles  $CBP$ ,  $C'BP$  are also equal, Euc. I. 13: next by Euc. I. 4,  $CP$ ,  $PC'$  are proved equal; lastly prove  $CC'$  to be equal to  $CP$  or  $PC'$ .

55. In the fig. Euc. I. 1, produce  $AB$  both ways to meet the circles in  $D$  and  $E$ , join  $CD$ ,  $CE$ , then  $CDE$  is an isosceles triangle, having each of the angles at the base one-fourth of the angle at the vertex. At  $E$  draw  $EG$  perpendicular to  $DB$  and meeting  $DC$  produced in  $G$ . Then  $CEG$  is an equilateral triangle.

56. Join  $CC'$ , and shew that the angles  $CC'F$ ,  $CC'G$  are equal to two right angles; also that the line  $FC'G$  is equal to the diameter.

57. Construct the figure and by Euc. I. 32. If the angle  $BAC$  be a right angle, then the angle  $BDC$  is half a right angle.

58. Let the lines which bisect the three exterior angles of the triangle  $ABC$  form a new triangle  $A'B'C'$ . Then each of the angles at  $A'$ ,  $B'$ ,  $C'$  may be shewn to be equal to half of the angles at  $A$  and  $B$ ,  $B$  and  $C$ ,  $C$  and  $A$  respectively. And it will be found that half the sums of every two of three unequal numbers whose sum is constant, have less differences than the three numbers themselves.

59. The first case may be shewn by Euc. I. 4: and the second by Euc. I. 32, 6, 15.

60. At  $D$  any point in a line  $EF$ , draw  $DC$  perpendicular to  $EF$  and equal to the given perpendicular on the hypotenuse. With centre  $C$  and radius equal to the given base describe a circle cutting  $EF$  in  $B$ . At  $C$  draw  $CA$  perpendicular to  $CB$  and meeting  $EF$  in  $A$ . Then  $ABC$  is the triangle required.

61. Let  $ABC$  be the required triangle having the angle  $ACB$  a right angle. In  $BC$  produced, take  $CE$  equal to  $AC$ , and with center  $B$  and radius  $BA$  describe a circular arc cutting  $CE$  in  $D$ , and join  $AD$ . Then  $DE$  is the difference between the sum of the two sides  $AC$ ,  $CB$  and the hypotenuse  $AB$ ; also one side  $AC$  the perpendicular is given. Hence the construction. On any line  $EB$  take  $EC$  equal to the given side,  $ED$  equal to the given difference. At  $C$ , draw  $CA$  perpendicular to  $CB$ , and equal to  $EC$ , join  $AD$ , at  $A$  in  $AD$  make the angle  $DAB$  equal to  $ADB$ , and let  $AB$  meet  $EB$  in  $B$ . Then  $ABC$  is the triangle required.

62. (1) Let  $ABC$  be the triangle required, having  $ACB$  the right angle. Produce  $AB$  to  $D$  making  $AD$  equal to  $AC$  or  $CB$ : then  $BD$  is the sum of the sides. Join  $DC$ : then the angle  $ADC$  is one-fourth of a right angle, and  $DBC$  is one-half of a right angle. Hence to construct: at  $B$  in  $BD$  make the angle  $DBM$  equal to half a right angle, and at  $D$  the angle  $BDC$  equal to one-fourth of a right angle, and let  $DC$  meet  $BM$  in  $C$ . At  $C$  draw  $CA$  at right angles to  $BC$  meeting  $BD$  in  $A$ : and  $ABC$  is the triangle required.

(2) Let  $ABC$  be the triangle,  $C$  the right angle: from  $AB$  cut off  $AD$  equal to  $AC$ ; then  $BD$  is the difference of the hypotenuse and one side. Join  $CD$ ; then the angles  $ACD$ ,  $ADC$  are equal, and each is half the supplement of  $DAC$ , which is half a right angle. Hence the construction.

63. Take any straight line terminated at  $A$ . Make  $AB$  equal to the difference of the sides, and  $AC$  equal to the hypotenuse. At  $B$  make the angle  $CBD$  equal to half a right angle, and with center  $A$  and radius  $AC$  describe a circle cutting  $BD$  in  $D$ : join  $AD$ , and draw  $DE$  perpendicular to  $AC$ . Then  $ADE$  is the required triangle.



64. Let  $BC$  the given base be bisected in  $D$ . At  $D$  draw  $DE$  at right angles to  $BC$  and equal to the sum of one side of the triangle and the perpendicular from the vertex on the base: join  $DB$ , and at  $B$  in  $BE$  make the angle  $EBA$  equal to the angle  $BED$ , and let  $BA$  meet  $DE$  in  $A$ : join  $AC$ , and  $ABC$  is the isosceles triangle.

65. This construction may be effected by means of Prob. 4, p. 71.

66. The perpendicular from the vertex on the base of an equilateral triangle bisects the angle at the vertex which is two-thirds of one right angle.

67. Let  $ABC$  be the equilateral triangle of which a side is required to be found, having given  $BD$ ,  $CD$  the lines bisecting the angles at  $B$ ,  $C$ . Since the angles  $DBC$ ,  $DCB$  are equal, each being one-third of a right angle, the sides  $BD$ ,  $DC$  are equal, and  $BDC$  is an isosceles triangle having the angle at the vertex the supplement of a third of two right angles. Hence the side  $BC$  may be found.

68. Let the given angle be taken, (1) as the *included angle* between the given sides; and (2) as the *opposite angle* to one of the given sides. In the latter case, an ambiguity will arise if the angle be an acute angle, and opposite to the less of the two given sides.

69. Let  $ABC$  be the required triangle,  $BC$  the given base,  $CD$  the given difference of the sides  $AB$ ,  $AC$ : join  $BD$ , then  $DBC$  by Euc. I. 18, can be shewn to be half the difference of the angles at the base, and  $AB$  is equal to  $AD$ . Hence at  $B$  in the given base  $BC$ , make the angle  $CBD$  equal to half the difference of the angles at the base. On  $CB$  take  $CE$  equal to the difference of the sides, and with center  $C$  and radius  $CE$ , describe a circle cutting  $BD$  in  $D$ : join  $CD$  and produce it to  $A$ , making  $DA$  equal to  $DB$ . Then  $ABC$  is the triangle required.

70. On the line which is equal to the perimeter of the required triangle describe a triangle having its angles equal to the given angles. Then bisect the angles at the base; and from the point where these lines meet, draw lines parallel to the sides and meeting the base.

71. Let  $ABC$  be the required triangle,  $BC$  the given base, and the side  $AB$  greater than  $AC$ . Make  $AD$  equal to  $AC$ , and draw  $CD$ . Then the angle  $BCD$  may be shewn to be equal to half the difference, and the angle  $DCA$  equal to half the sum of the angles at the base. Hence  $ABC$ ,  $ACB$  the angles at the base of the triangle are known.

72. Let the two given lines meet in  $A$ , and let  $B$  be the given point. If  $BC$ ,  $BD$  be supposed to be drawn making equal angles with  $AC$ , and if  $AD$  and  $DC$  be joined,  $BCD$  is the triangle required, and the figure  $ACBD$  may be shewn to be a parallelogram. Whence the construction.

73. It can be shewn that lines drawn from the angles of a triangle to bisect the opposite sides, intersect each other at a point which is two-thirds of their lengths from the angular points from which they are drawn. Let  $ABC$  be the triangle required,  $AD$ ,  $BE$ ,  $CF$  the given lines from the angles drawn to the bisections of the opposite sides and intersecting in  $G$ . Produce  $GD$ , making  $DH$  equal to  $DG$ , and join  $BH$ ,  $CH$ : the figure  $GBHC$  is a parallelogram. Hence the construction.

74. Let  $ABC$  (fig. to Euc. I. 20.) be the required triangle, having the base  $BC$  equal to the given base, the angle  $ABC$  equal to the given angle, and the two sides  $BA$ ,  $AC$  together equal to the given line  $BD$ . Join  $DC$ , then since  $AD$  is equal to  $AC$ , the triangle  $ACD$  is isosceles, and therefore the angle  $ADC$  is equal to the angle  $ACD$ . Hence the construction.

75. Let  $ABC$  be the required triangle (fig. to Euc. I. 18), having the angle  $ACB$  equal to the given angle, and the base  $BC$  equal to the given

line, also  $CD$  equal to the difference of the two sides  $AB$ ,  $AC$ . If  $BD$  be joined, then  $ABD$  is an isosceles triangle. Hence the synthesis. Does this construction hold good in all cases?

76. Let  $ABC$  be the required triangle, (fig. Euc. i. 18), of which the side  $BC$  is given and the angle  $BAC$ , also  $CD$  the difference between the sides  $AB$ ,  $AC$ . Join  $BD$ ; then  $AB$  is equal to  $AD$ , because  $CD$  is their difference, and the triangle  $ABD$  is isosceles, whence the angle  $ABD$  is equal to the angle  $ADB$ ; and since  $BAD$  and twice the angle  $ABD$  are equal to two right angles, it follows that  $ABD$  is half the supplement of the given angle  $BAC$ . Hence the construction of the triangle.

77. Let  $AB$  be the given base: at  $A$  draw the line  $AD$  to which the line bisecting the vertical angle is to be parallel. At  $B$  draw  $BE$  parallel to  $AD$ ; from  $A$  draw  $AE$  equal to the given sum of the two sides to meet  $BE$  in  $E$ . At  $B$  make the angle  $EBC$  equal to the angle  $BEA$ , and draw  $CF$  parallel to  $AD$ . Then  $ACB$  is the triangle required.

78. Take any point in the given line, and apply Euc. i. 23, 31.

79. On one of the parallel lines take  $EF$  equal to the given line, and with center  $E$  and radius  $EF$  describe a circle cutting the other in  $G$ . Join  $EG$ , and through  $A$  draw  $ABC$  parallel to  $EG$ .

80. This will appear from Euc. i. 29, 15, 26.

81. Let  $AB$ ,  $AC$ ,  $AD$ , be the three lines. Take any point  $E$  in  $AC$ , and on  $EC$  make  $EF$  equal to  $EA$ , through  $F$  draw  $FG$  parallel to  $AB$ , join  $GE$  and produce it to meet  $AB$  in  $H$ . Then  $GE$  is equal to  $GH$ .

82. Apply Euc. i. 32, 29.

83. From  $E$  draw  $EG$  perpendicular on the base of the triangle, then  $ED$  and  $EF$  may each be proved equal to  $EG$ , and the figure shewn to be equilateral. Three of the angles of the figure are right angles.

84. The greatest parallelogram which can be constructed with given sides can be proved to be rectangular.

85. Let  $AB$  be one of the diagonals: at  $A$  in  $AB$  make the angle  $BAC$  less than the required angle, and at  $A$  in  $AC$  make the angle  $CAD$  equal to the required angle. Bisect  $AB$  in  $E$  and with center  $E$  and radius equal to half the other diagonal describe a circle cutting  $AC$ ,  $AD$  in  $F$ ,  $G$ . Join  $FB$ ,  $BG$ : then  $AFBG$  is the parallelogram required.

86. This problem is the same as the following; having given the base of a triangle, the vertical angle and the sum of the sides, to construct the triangle. This triangle is one half of the required parallelogram.

87. Draw a line  $AB$  equal to the given diagonal, and at the point  $A$  make an angle  $BAC$  equal to the given angle. Bisect  $AB$  in  $D$ , and through  $D$  draw a line parallel to the given line and meeting  $AC$  in  $C$ . This will be the position of the other diagonal. Through  $B$  draw  $BE$  parallel to  $CA$ , meeting  $CD$  produced in  $E$ ; join  $AE$ , and  $BC$ . Then  $ACBE$  is the parallelogram required.

88. Construct the figures and by Euc. i. 24.

89. By Euc. i. 4, the opposite sides may be proved to be equal.

90. Let  $ABCD$  be the given parallelogram; construct the other parallelogram  $A'B'C'D'$  by drawing the lines required, also the diagonals  $AC$ ,  $A'C'$ , and shew that the triangles  $ABC$ ,  $A'B'C'$  are equiangular.

91.  $A'D'$  and  $B'C'$  may be proved to be parallel.

92. Apply Euc. i. 29, 32.

93. The points  $D$ ,  $D'$ , are the intersections of the diagonals of two rectangles: if the rectangles be completed, and the lines  $OD$ ,  $OD'$  be produced, they will be the other two diagonals.

94. Let the line drawn from  $A$  fall without the parallelogram, and

let  $CC'$ ,  $BB'$ ,  $DD'$ , be the perpendiculars from  $C$ ,  $B$ ,  $D$ , on the line drawn from  $A$ ; from  $B$  draw  $BE$  parallel to  $AC'$ , and the truth is manifest. Next, let the line from  $A$  be drawn so as to fall within the parallelogram.

95. Let the diagonals intersect in  $E$ . In the triangles  $DCB$ ,  $CDA$ , two angles in each are respectively equal and one side  $DE$ : wherefore the diagonals  $DB$ ,  $AC$  are equal: also since  $DE$ ,  $EC$  are equal, it follows that  $EA$ ,  $EB$  are equal. Hence  $DEC$ ,  $AEB$  are two isosceles triangles having their vertical angles equal, wherefore the angles at their bases are equal respectively, and therefore the angle  $CDB$  is equal to  $DBA$ .

96. (1) By supposing the point  $P$  found in the side  $AB$  of the parallelogram  $ABCD$ , such that the angle contained by  $AP$ ,  $PC$  may be bisected by the line  $PD$ ;  $CP$  may be proved equal to  $CD$ ; hence the solution.

(2) By supposing the point  $P$  found in the side  $AB$  produced, so that  $PD$  may bisect the angle contained by  $ABP$  and  $PC$ ; it may be shewn that the side  $AB$  must be produced, so that  $BP$  is equal to  $BD$ .

97. This may be shewn by *Euc. i. 35*.

98. Let  $D$ ,  $E$ ,  $F$  be the bisections of the sides  $AB$ ,  $BC$ ,  $CA$  of the triangle  $ABC$ : draw  $DE$ ,  $EF$ ,  $FD$ ; the triangle  $DEF$  is one-fourth of the triangle  $ABC$ . The triangles  $DBE$ ,  $FBE$  are equal, each being one-fourth of the triangle  $ABC$ :  $DF$  is therefore parallel to  $BE$ , and  $DBEF$  is a parallelogram of which  $DE$  is a diagonal.

99. This may be proved by applying *Euc. i. 38*.

100. Apply *Euc. i. 37, 38*.

101. On any side  $BC$  of the given triangle  $ABC$ , take  $BD$  equal to the given base; join  $AD$ , through  $C$  draw  $CE$  parallel to  $AD$ , meeting  $BA$  produced if necessary in  $E$ , join  $ED$ ; then  $BDE$  is the triangle required. By a process somewhat similar the triangle may be formed when the altitude is given.

102. Apply the preceding problem (101) to make a triangle equal to one of the given triangles and of the same altitude as the other given triangle. Then the sum or difference can be readily found.

103. First construct a triangle on the given base equal to the given triangle; next form an isosceles triangle on the same base equal to this triangle.

104. Make an isosceles triangle equal to the given triangle, and then this isosceles triangle into an equal equilateral triangle.

105. Make a triangle equal to the given parallelogram upon the given line, and then a triangle equal to this triangle, having an angle equal to the given angle.

106. If the figure  $ABCD$  be one of four sides; join the opposite angles  $A$ ,  $C$  of the figure, through  $D$  draw  $DE$  parallel to  $AC$  meeting  $BC$  produced in  $E$ , join  $AE$ :—the triangle  $ABE$  is equal to the four-sided figure  $ABCD$ .

If the figure  $ABCDE$  be one of five sides, produce the base both ways, and the figure may be transformed into a triangle, by two constructions similar to that employed for a figure of four sides. If the figure consists of six, seven, or any number of sides, the same process must be repeated.

107. Draw two lines from the bisection of the base parallel to the two sides of the triangle.

108. This may be shewn *ex absurdo*.

109. On the same base  $AB$ , and on the same side of it, let two triangles  $ABC$ ,  $ABD$  be constructed, having the side  $BD$  equal to  $BC$ , the angle  $ABC$  a right angle, but the angle  $ABD$  not a right angle; then the triangle  $ABC$  is greater than  $ABD$ , whether the angle  $ABD$  be acute or obtuse.

110. Let  $ABC$  be a triangle whose vertical angle is  $A$ , and whose

base BC is bisected in D; let any line EDG be drawn through D, meeting AC the greater side in G and AB produced in E, and forming a triangle AEG having the same vertical angle A. Draw BH parallel to AC, and the triangles BDH, GDC are equal. Euc. I. 26.

111. Let two triangles be constructed on the same base with equal perimeters, of which one is isosceles. Through the vertex of that which is not isosceles draw a line parallel to the base, and intersecting the perpendicular drawn from the vertex of the isosceles triangle upon the common base. Join this point of intersection and the extremities of the base.

112. (1) DF bisects the triangle ABC (fig. Prop. 6, p. 73.) On each side of the point F in the line BC, take FG, FH, each equal to one-third of BF, the lines DG, DH shall trisect the triangle. Or,

Let ABC be any triangle, D the given point in BC. Trisect BC in E, F. Join AD, and draw EG, FH parallel to AD. Join DG, DH; these lines trisect the triangle. Draw AE, AF and the proof is manifest.

(2) Let ABC be any triangle; trisect the base BC in D, E, and join AD, AE. From D, E, draw DP, EP parallel to AB, AC and meeting in P. Join AP, BP, CP; these three lines trisect the triangle.

(3) Let P be the given point within the triangle ABC. Trisect the base BC in D, E. From the vertex A draw AD, AE, AP. Join PD, draw AG parallel to PD and join PG. Then BGPA is one-third of the triangle. The problem may be solved by trisecting either of the other two sides and making a similar construction.

113. The base may be divided into nine equal parts, and lines may be drawn from the vertex to the points of division. Or, the sides of the triangle may be trisected, and the points of trisection joined.

114. It is proved, Euc. I. 34, that each of the diagonals of a parallelogram bisects the figure, and it may be shewn that they also bisect each other. It is hence manifest that any straight line, whatever may be its position, which bisects a parallelogram, *must* pass through the intersection of the diagonals.

115. See the remark on the preceding problem 114.

116. Trisect the side AB in E, F, and draw EG, FH parallel to AD or BC, meeting DC in G and H. If the given point P be in EF, the two lines drawn from P through the bisections of EG and FH will trisect the parallelogram. If P be in FB, a line from P through the bisection of FH will cut off one-third of the parallelogram, and the remaining trapezium is to be bisected by a line from P, one of its angles. If P coincide with E or F, the solution is obvious.

117. Construct a right-angled parallelogram by Euc. I. 44, equal to the given quadrilateral figure, and from one of the angles, draw a line to meet the opposite side and equal to the base of the rectangle, and a line from the adjacent angle parallel to this line will complete the rhombus.

118. Bisect BC in D, and through the vertex A, draw AE parallel to BC, with center D and radius equal to half the sum of AB, AC, describe a circle cutting AE in E.

119. Produce one side of the square till it becomes equal to the diagonal, the line drawn from the extremity of this produced side and parallel to the adjacent side of the square, and meeting the diagonal produced, determines the point required.

120. Let fall upon the diagonal perpendiculars from the opposite angles of the parallelogram. These perpendiculars are equal, and each pair of titules. If the point be not on the diagonal, draw through the given point, a line parallel to a side of the parallelogram.

121. One case is included in Theo. 120. The other case, when the point is in the diagonal produced, is obvious from the same principle.

122. The triangles DCF, ABF may be proved to be equal to half of the parallelogram by Euc. i. 41.

123. Apply Euc. i. 41, 38.

124. If a line be drawn parallel to AD through the point of intersection of the diagonal, and the line drawn through O parallel to AB; then by Euc. i. 43, 41, the truth of the theorem is manifest.

125. It may be remarked that parallelograms, are divided into pairs of equal triangles by the diagonals, and therefore by taking the triangle ABD equal to the triangle ABC, the property may be easily shewn.

126. The triangle ABD is one half of the parallelogram ABCD, Euc. i. 34. And the triangle DKC is one half of the parallelogram CDHG, Euc. i. 41, also for the same reason the triangle AKB is one half of the parallelogram AHGB: therefore the two triangles DKC, AKB are together one half of the whole parallelogram ABCD. Hence the two triangles DKC, AKB are equal to the triangle ABD: take from these equals the equal parts which are common, therefore the triangle CKF is equal to the triangles AHK, KBD: wherefore also taking AHK from these equals, then the difference of the triangles CKF, AHK is equal to the triangle KBC: and the doubles of these are equal, or the difference of the parallelograms CFKG, AHKE is equal to twice the triangle KBD.

127. First prove that the perimeter of a square is less than the perimeter of an equal rectangle: next, that the perimeter of the rectangle is less than the perimeter of any other equal parallelogram.

128. This may be proved by shewing that the area of the isosceles triangle is greater than the area of any other triangle which has the same vertical angle, and the sum of the sides containing that angle is equal to the sum of the equal sides of the isosceles triangle.

129. Let ABC be an isosceles triangle (fig. Euc. i. 42), AE perpendicular to the base BC, and AECG the equivalent rectangle. Then AC is greater than AE, &c.

130. Let the diagonal AC bisect the quadrilateral figure ABCD. Bisect AC in E, join BE, ED, and prove BE, ED in the same straight line and equal to one another.

131. Apply Euc. i. 15.

132. Apply Euc. i. 20.

133. This may be shewn by Euc. i. 20.

134. Let AB be the longest and CD the shortest side of the rectangular figure. Produce AD, BC to meet in E. Then by Euc. i. 32.

135. Let ABCD be the quadrilateral figure, and E, F, two points in the opposite sides AB, CD, join EF and bisect it in G; and through G draw a straight line HGK terminated by the sides AD, BC; and bisected in the point G. Then EF, HK are the diagonals of the required parallelogram.

136. After constructing the figure, the proof offers no difficulty.

137. If any line be assumed as a diagonal, if the four given lines taken two and two be always greater than this diagonal, a four-sided figure may be constructed having the assumed line as one of its diagonals: and it may be shewn that when the quadrilateral is possible, the sum of every three given sides is greater than the fourth.

138. Draw the two diagonals, then four triangles are formed, two on one side of each diagonal. Then two of the lines drawn through the points of bisection of two sides may be proved parallel to one diagonal, and two

parallel to the other diagonal, in the same way as Theo. 97, *supra*. The other property is manifest from the relation of the areas of the triangles made by the lines drawn through the bisections of the sides.

139. It is sufficient to suggest, that triangles on equal bases, and of equal altitudes, are equal.

140. Let the side AB be parallel to CD, and let AB be bisected in E and CD in F, and let EF be drawn. Join AF, BF, then Euc. i. 38.

141. Let BCED be a trapezium of which DC, BE are the diagonals intersecting each other in G. If the triangle DBG be equal to the triangle EGC, the side DE may be proved parallel to the side BC, by Euc. i. 39.

142. Let ABCD be the quadrilateral figure having the sides AB, CD, parallel to one another, and AD, BC equal. Through B draw BE parallel to AD, then ABED is a parallelogram.

143. Let ABCD be the quadrilateral having the side AB parallel to CD. Let E, F be the points of bisection of the diagonals BD, AC, and join EF and produce it to meet the sides AD, BC in G and H. Through H draw LHK parallel to DA meeting DC in L and AB produced in K. Then BK is half the difference of DC and AB.

144. (1) Reduce the trapezium ABCD to a triangle BAE by Prob. 106, *supra*, and bisect the triangle BAE by a line AF from the vertex. If F fall without BC, through F draw FG parallel to AC or DE, and join AG.

Or thus. Draw the diagonals AC, BD: bisect BD in E, and join AE, EC. Draw FEG parallel to AC the other diagonal, meeting AD in F, and DC in G. AG being joined, bisects the trapezium.

(2) Let E be the given point in the side AD. Join EB. Bisect the quadrilateral EBCD by EF. Make the triangle EFG equal to the triangle EAB, on the same side of EF as the triangle AB. Bisect the triangle EFG by EH. EH bisects the figure.

145. If a straight line be drawn from the given point through the intersection of the diagonals and meeting the opposite side of the square; the problem is then reduced to the bisection of a trapezium by a line drawn from one of its angles.

146. If the four sides of the figure be of different lengths, the truth of the theorem may be shewn. If, however, two adjacent sides of the figure be equal to one another, as also the other two, the lines drawn from the angles to the bisection of the longer diagonal, will be found to divide the trapezium into four triangles which are equal in area to one another. Euc. i. 38.

147. Apply Euc. i. 47, observing that the shortest side is one half of the longest.

148. Find by Euc. i. 47, a line the square on which shall be seven times the square on the given line. Then the triangle which has these two lines containing the right angle shall be the triangle required.

149. Apply Euc. i. 47.

150. Let the base BC be bisected in D, and DE be drawn perpendicular to the hypotenuse AC. Join AD: then Euc. i. 47.

151. Construct the figure, and the truth is obvious from Euc. i. 47.

152. See Theo. 32, p. 76, and apply Euc. i. 47.

153. Draw the lines required and apply Euc. i. 47.

154. Apply Euc. i. 47.

155. Apply Euc. i. 47.

156. Apply Euc. i. 47, observing that the square on any line is four times the square on half the line.

157. Apply Euc. I. 47, to express the squares of the three sides in terms of the squares on the perpendiculars and on the segments of AB.

158. By Euc. I. 47. bearing in mind that the square described on any line is four times the square described upon half the line.

159. The former part is at once manifest by Euc. I. 47. Let the diagonals of the square be drawn, and the given point be supposed to coincide with the intersection of the diagonals, the minimum is obvious. Find its value in terms of the side.

160. (a) This is obvious from Euc. I. 13.

(b) Apply Euc. I. 32, 29.

(c) Apply Euc. I. 5, 29.

(d) Let AL meet the base BC in P, and let the perpendiculars from F, K meet BC produced in M and N respectively; then the triangles  $\triangle APB$ ,  $\triangle FMB$  may be proved to be equal in all respects, as also  $\triangle APC$ ,  $\triangle KNC$ .

(e) Let fall DQ perpendicular on FB produced. Then the triangle  $\triangle QDB$  may be proved equal to each of the triangles  $\triangle ABC$ ,  $\triangle DBF$ ; whence the triangle  $\triangle DBF$  is equal to the triangle  $\triangle ABC$ .

Perhaps however the better method is to prove at once that the triangles  $\triangle ABC$ ,  $\triangle FBD$  are equal, by shewing that they have two sides equal in each triangle, and the included angles, one the supplement of the other.

(f) If DQ be drawn perpendicular on FB produced, FQ may be proved to be bisected in the point B, and DQ equal to AC. Then the square on FD is found by the right-angled triangle  $\triangle FQD$ . Similarly, the square on KE is found, and the sum of the squares on FD, EK, GH will be found to be six times the square on the hypotenuse.

(g) Through A draw PAQ parallel to BC and meeting DB, EC produced in P, Q. Then by the right angled triangles.

161. Let any parallelograms be described on any two sides AB, AC of a triangle ABC, and the sides parallel to AB, AC be produced to meet in a point P. Join PA. Then on either side of the base BC, let a parallelogram be described having two sides equal and parallel to AP. Produce AP and it will divide the parallelogram on BC into two parts respectively equal to the parallelograms on the sides. Euc. I. 35, 36.

162. Let the equilateral triangles ABD, BCE, CAF be described on AB, BC, CA, the sides of the triangle ABC having the right angle at A.

Join DC, AK: then the triangles DBC, ABE are equal. Next draw DG perpendicular to AB and join CG: then the triangles BDG, DAG, DGC are equal to one another. Also draw AH, EK perpendicular to BC; the triangles EKH, EKA are equal. Whence may be shewn that the triangle ABD is equal to the triangle BHE, and in a similar way may be shewn that CAF is equal to CHE.

The restriction is unnecessary: it only brings AD, AE into the same line.

## GEOMETRICAL EXERCISES ON BOOK II.

### HINTS, &c.

6. See the figure Euc. II. 5.

7. This Problem is equivalent to the following: construct an isosceles right-angled triangle, having given one of the sides which contains the right angle.

8. In the question for E read D. Construct the square on AB, and the property is obvious.

9. The sum of the squares on the two parts of any line is least when the two parts are equal.

10. A line may be found the square on which is double the square on the given line. The problem is then reduced to:—having given the hypotenuse and the sum of the sides of a right-angled triangle, construct the triangle.

11. This follows from Euc. II. 5, Cor.

12. This problem is, in other words, Given the sum of two lines and the sum of their squares, to find the lines. Let AB be the given straight line, at B draw BC at right angles to AB, bisect the angle ABC by BD. On AB take AE equal to the side of the given square, and with center A and radius AE describe a circle cutting BD in D, from D draw DF perpendicular to AB, the line AB is divided in F as was required.

13. Let AB be the given line. Produce AB to C making BC equal to three times the square on AB. From BA cut off BD equal to BC; then D is the point of section such that the squares on AB and BD are double of the square on AD.

14. In the fig. Euc. II. 7. Join BF, and draw FL perpendicular on GD. Half the rectangle DB, BG, may be proved equal to the rectangle AB, BC.

Or, join KA, CD, KD, CK. Then CK is perpendicular to BD. And the triangles CBD, KBD are each equal to the triangle ABK. Hence, twice the triangle ABK is equal to the figure CBKD; but twice the triangle ABK is equal to the rectangle AB, BC; and the figure CBKD is equal to half the rectangle DB and CK, the diagonals of the squares on AB, BC. Wherefore, &c.

15. The difference between the two unequal parts may be shewn to be equal to twice the line between the points of section.

16. This proposition is only another form of stating Euc. II. 7.

17. In the figure, Theo. 7, p. 69, draw PQ, PR, PS perpendiculars on AB, AD, AC respectively: then since the triangle PAC is equal to the two triangles PAB, PAD, it follows that the rectangle contained by PS, AC, is equal to the sum of the rectangles PQ, AB, and PR, AD. When is the rectangle PS, AC equal to the difference of the other two rectangles?

18. Through E draw EG parallel to AB, and through F, draw FHK parallel to BC and cutting EG in H. Then the area of the rectangle is made up of the areas of four triangles; whence it may be readily shewn that twice the area of the triangle AFE, and the figure AGHK is equal to the area ABCD.

19. Apply Euc. II. 11.

20. The vertical angles at L may be proved to be equal, and each of them a right angle.

21. Apply Euc. II. 4, II. 1. 47.

22. Produce FG, DB to meet in L, and draw the other diagonal LHC, which passes through H, because the complements AG, BK are equal. Then LH may be shewn to be equal to Ff, and to Dd.

23. The common intersection of the three lines divides each into two parts, one of which is double of the other, and this point is the vertex of three triangles which have lines drawn from it to the bisection of the bases. Apply Euc. II. 12, 13.

24. Apply Theorem 3, p. 104, and Euc. I. 47.

25. This will be found to be that particular case of Euc. II. 12, in which the distance of the obtuse angle from the foot of the perpendicular,



is half of the side subtended by the right angle made by the perpendicular and the base produced.

26. (1) Let the triangle be acute-angled, (Euc. II. 13, fig. 1.)

Let AC be bisected in E, and BE be joined; also EF be drawn perpendicular to BC. EF is equal to FC. Then the square on BE may be proved to be equal to the square on BC and the rectangle BD, BC.

(2) If the triangle be obtuse-angled, the perpendicular EF falls *within* or *without* the base according as the bisecting line is drawn from the *obtuse* or the *acute* angle at the base.

27. This may be shewn from theorem 3, p. 114.

28. Let the perpendicular AD be drawn from A on the base BC. It may be shewn that the base BC must be produced to a point E, such that CE is equal to the difference of the segments of the base made by the perpendicular.

29. Since the base and area are given, the altitude of the triangle is known. Hence the problem is reduced to;—Given the base and altitude of a triangle, and the line drawn from the vertex to the bisection of the base, construct the triangle.

30. This follows immediately from Euc. I. 47.

31. Apply Euc. II. 13.

32. The truth of this property depends on the fact that the rectangle contained by AC, CB is equal to that contained by AB, CD.

33. Let P the required point in the base AB be supposed to be known. Join CP. It may then be shewn that the property stated in the Problem is contained in Theorem 3, p. 114.

34. This may be shewn from Euc. I. 47; II. 5. Cor.

35. From C let fall CF perpendicular on AB. Then ACE is an obtuse-angled, and BEC an acute-angled triangle. Apply Euc. II. 12, 13; and by Euc. I. 47, the squares on AC and CB are equal to the square on AB.

36. Apply Euc. I. 47, II. 4; and the note p. 102, on Euc. II. 4.

37. Draw a perpendicular from the vertex to the base, and apply Euc. I. 47; II. 5, Cor. Enunciate and prove the proposition when the straight line drawn from the vertex meets the base produced.

38. This follows directly from Euc. II. 13, Case 1.

39. The truth of this proposition may be shewn from Euc. I. 47; II. 4.

40. Let the square on the base of the isosceles triangle be described. Draw the diagonals of the square, and the proof is obvious.

41. Let ABC be the triangle required, such that the square on AB is three times the square on AC or BC. Produce BC and draw AD perpendicular to BC. Then by Euc. II. 12, CD may be shewn to be equal to one half of BC. Hence the construction.

42. Apply Euc. II. 12, and Theorem 38, p. 118.

43. Draw EF parallel to AB and meeting the base in F; draw also EG perpendicular to the base. Then by Euc. I. 47; II. 5, Cor.

44. Bisect the angle B by BD meeting the opposite side in D, and draw BE perpendicular to AC. Then by Euc. I. 47; II. 5, Cor.

45. This follows directly from Theorem 3, p. 114.

46. Draw the diagonals intersecting each other in P, and join OP. By Theo. 3, p. 114.

47. Draw from any two opposite angles, straight lines to meet in the bisection of the diagonal joining the other angles. Then by Euc. II. 12, 13.

48. Draw two lines from the point of bisection of either of the bisected sides to the extremities of the opposite side; and three triangles will be formed, two on one of the bisected sides and one on the other, in

each of which is a line drawn from the vertex to the bisection of the base. Then by Theo. 3, p, 114.

49. If the extremities of the two lines which bisect the opposite sides of the trapezium be joined, the figure formed is a parallelogram which has its sides respectively parallel to, and equal to, half the diagonals of the trapezium. The sum of the squares on the two diagonals of the trapezium may be easily shewn to be equal to the sum of the squares on the four sides of the parallelogram.

50. Draw perpendiculars from the extremities of one of the parallel sides, meeting the other side produced, if necessary. Then from the four right-angled triangles thus formed, may be shewn the truth of the proposition.

51. In the problem, for triangle read rectangle. Let ABCD be any trapezium having the side AD parallel to BC. Draw the diagonal AC, then the sum of the triangles ABC, ADC may be shewn to be equal to the rectangle contained by the altitude and half the sum of AD and BC.

52. Let ABCD be the trapezium, having the sides AB, CD, parallel, and AD, BC equal. Join AC and draw AE perpendicular to DC. Then by Euc. II. 13.

53. Let ABC be any triangle; AHKB, AGFC, BDEC, the squares upon their sides; EF, GH, KL the lines joining the angles of the squares. Produce GA, KB, EC, and draw HN, DQ, FR perpendiculars upon them respectively: also draw AP, BM, CS perpendiculars on the sides of the triangle. Then AN may be proved to be equal to AM; CR to CP; and BQ to BS; and by Euc. II. 12, 13.

54. Convert the triangle into a rectangle, then Euc. II. 14.

55. Find a rectangle equal to the two figures, and apply Euc. II. 14.

56. Find the side of a square which shall be equal to the given rectangle. See Prob. I. p. 113.

57. On any line PQ take AB equal to the given difference of the sides of the rectangle, at A draw AC at right angles to AB, and equal to the side of the given square; bisect AB in O and join OC; with center O and radius OC describe a semicircle meeting PQ in D and E. Then the lines AD, AE have AB for their difference, and the rectangle contained by them is equal to the square on AC.

58. Apply Euc. II. 14.

## GEOMETRICAL EXERCISES ON BOOK III.

### HINTS, &c.

7. Euc. III. 3, suggests the construction.

8. The given point may be either within or without the circle. Find the center of the circle, and join the given point and the center, and upon this line describe a semicircle, a line equal to the given distance may be drawn from the given point to meet the arc of the semicircle. When the point is without the circle, the given distance may meet the diameter produced.

9. This may be easily shewn to be a straight line passing through the center of the circle.

10. The two chords form by their intersections the sides of two isosceles triangles, of which the parallel chords in the circle are the bases.

11. The angles in equal segments are equal, and by Euc. I. 29. If the chords are equally distant from the center, the lines intersect the diameter in the center of the circle.

12. Construct the figure and the arc BC may be proved equal to the arc B'C'.

13. The point determined by the lines drawn from the bisections of the chords and at right angles to them respectively, will be the center of the required circle.

14. Construct the figures: the proof offers no difficulty.

15. On any radius construct an isosceles right-angled triangle, and produce the side which meets the circumference.

16. Join the extremities of the chords, then Euc. I. 27; III. 28.

17. Take the center O, and join AP, AO, &c. and apply Euc. I. 20.

18. Draw any straight line intersecting two parallel chords and meeting the circumference.

19. Produce the radii to meet the circumference.

20. Join AD, and the first equality follows directly from Euc. III. 20, I. 32. Also by joining AC, the second equality may be proved in a similar way. If however the line AD do not fall on the same side of the center O as E, it will be found that the *difference*, not the *sum* of the two angles, is equal to 2. AED. See note to Euc. III. 20, p. 155.

21. Let DKE, DBO (fig. Euc. III. 8) be two lines equally inclined to DA, then KE may be proved to be equal to BO, and the segments cut off by equal straight lines in the same circle, as well as in equal circles, are equal to one another.

22. Apply Euc. I. 15, and III. 21.

23. This is the same as Euc. III. 34, with the condition, that the line must pass through a given point.

24. Let the segments AHB, AKC be externally described on the given lines AB, AC, to contain angles equal to BAC. Then by the converse to Euc. III. 32, AB touches the circle AKC, and AC the circle AHB.

25. Let ABC be a triangle of which the base or longest side is BC, and let a segment of a circle be described on BC. Produce BA, CA to meet the arc of the segment in D, E, and join BD, CE. If circles be described about the triangles ABD, ACE, the sides AB, AC shall cut off segments similar to the segment described upon the base BC.

26. This is obvious from the note to Euc. III. 26, p. 156.

27. The segment must be described on the opposite side of the produced chord. By converse of Euc. III. 32.

28. If a circle be described upon the side AC as a diameter, the circumference will pass through the points D, E. Then Euc. III. 21.

29. Let AB, AC be the bounding radii, and D any point in the arc BC, and DE, DF, perpendiculars from D on AB, AC. The circle described on AD will always be of the same magnitude, and the angle EAF in it, is constant:—whence the arc EDF is constant, and therefore its chord EF.

30. Construct the figure, and let the circle with center O, described on AH as a diameter, intersect the given circle in P, Q, join OP, PE, and prove EP at right angles to OP.

31. If the tangent be required to be perpendicular to a given line: draw the diameter parallel to this line, and the tangent drawn at the extremity of this diameter will be perpendicular to the given line.

32. The straight line which joins the center and passes through the intersection of two tangents to a circle, bisects the angle contained by the tangents.

33. Draw two radii containing an angle equal to the supplement of the given angle; the tangents drawn at the extremities of these radii will contain the given angle.

34. Since the circle is to touch two parallel lines drawn from two given points in a third line, the radius of the circle is determined by the distance between the two given points.

35. It is sufficient to suggest that the angle between a chord and a tangent is equal to the angle in the alternate segment of the circle. Euc. III. 32.

36. Let AB be the given chord of the circle whose center is O. Draw DE touching the circle at any point E and equal to the given line; join DO, and with center O and radius DO describe a circle; produce the chord AB to meet the circumference of this circle in F: then F is the point required.

37. Let D be the point required in the diameter BA produced, such that the tangent DP is half of DB. Join CP, C being the center. Then CPD is a right-angled triangle, having the sum of the base PC and hypotenuse CD double of the perpendicular PD.

38. If BE intersect DF in K (fig. Euc. III. 37). Join FB, FE, then by means of the triangles, BE is shewn to be bisected in K at right angles.

39. Let AB, CD be any two diameters of a circle, O the center, and let the tangents at their extremities form the quadrilateral figure EFGH. Join EO, OF, then EO and OF may be proved to be in the same straight line, and similarly HO, OK.

NOTE.—This Proposition is equally true if AB, CD be any two chords whatever. It then becomes equivalent to the following proposition:—The diagonals of the circumscribed and inscribed quadrilaterals, intersect in the same point, the points of contact of the former being the angles of the latter figure.

40. Let C be the point without the circle from which the tangents CA, CB are drawn, and let DE be any diameter, also let AE, BD be joined, intersecting in P, then if CP be joined and produced to meet DE in G: CG is perpendicular to DE. Join DA, EB, and produce them to meet in F.

Then the angles DAE, EBD being angles in a semicircle, are right angles; or DB, EA are drawn perpendicular to the sides of the triangle DEF: whence the line drawn from F through P is perpendicular to the third side DE.

41. Let the chord AB, of which P is its middle point, be produced both ways to C, D, so that AC is equal to BD. From C, D, draw the tangents to the circle forming the tangential quadrilateral CKDR, the points of contact of the sides, being E, H, F, G. Let O be the center of the circle. Join EH, GF, CO, GO, FO, DO. Then EH and GF may be proved each parallel to CD, they are therefore parallel to one another. Whence is proved that both EF and DG bisect AB.

42. This is obvious from Euc. I. 29, and the note to III. 22. p. 156.

43. From any point A in the circumference, let any chord AB and tangent AC be drawn. Bisect the arc AB in D, and from D draw DE, DC perpendiculars on the chord AB and tangent AC. Join AD, the triangles ADE, ADC may be shewn to be equal.

44. Let A, B, be the given points. Join AB, and upon it describe a segment of a circle which shall contain an angle equal to the given angle. If the circle cut the given line, there will be two points; if it only touch the line, there will be one; and if it neither cut nor touch the line, the problem is impossible.

45. It may be shewn that the point required is determined by a perpendicular drawn from the center of the circle on the given line.

46. Let two lines AP, BP be drawn from the given points A, B, making equal angles with the tangent to the circle at the point of contact P, take any other point Q in the convex circumference, and join QA, QB: then by Prob. 4, p. 71, and Euc. 1. 21.

47. Let C be the center of the circle, and E the point of contact of DF with the circle. Join DC, CE, CF.

48. Let the tangents at E, F meet in a point R. Produce RE, RF to meet the diameter AB produced in S, T. Then RST is a triangle, and the quadrilateral RFOE may be circumscribed by a circle, and RPO may be proved to be one of the diagonals.

49. Let C be the middle point of the chord of contact: produce AC, BC to meet the circumference in B', A', and join AA', BB'.

50. Let A be the given point, and B the given point in the given line CD. At B draw BE at right angles to CD, join AB and bisect it in F, and from F draw FE perpendicular to AB and meeting BE in E. E is the center of the required circle.

51. Let O be the center of the given circle. Draw OA perpendicular to the given straight line; at O in OA make the angle AOP equal to the given angle, produce PO to meet the circumference again in Q. Then P, Q are two points from which tangents may be drawn fulfilling the required condition.

52. Let C be the center of the given circle, B the given point in the circumference, and A the other given point through which the required circle is to be made to pass. Join CB, the center of the circle is a point in CB produced. The center itself may be found in three ways.

53. Euc. III. 11 suggests the construction.

54. Let AB, AC be the two given lines which meet at A, and let D be the given point. Bisect the angle BAC by AE, the center of the circle is in AE. Through D draw DF perpendicular to AE, and produce DF to G, making FG equal to FD. Then DG is a chord of the circle, and the circle which passes through D and touches AB, will also pass through G and touch AC.

55. As the center is given, the line joining this point and center of the given circle, is perpendicular to that diameter, through the extremities of which the required circle is to pass.

56. Let AB be the given line and D the given point in it, through which the circle is required to pass, and AC the line which the circle is to touch. From D draw DE perpendicular to AB and meeting AC in C. Suppose O a point in AD to be the centre of the required circle. Draw OE perpendicular to AC, and join OC, then it may be shown that CO bisects the angle ACD.

57. Let the given circle be described. Draw a line through the center and intersection of the two lines. Next draw a chord perpendicular to this line, cutting off a segment containing the given angle. The circle described passing through one extremity of the chord and touching one of the straight lines, shall also pass through the other extremity of the chord and touch the other line.

58. The line drawn through the point of intersection of the two circles parallel to the line which joins their centers, may be shewn to be double of the line which joins their centers, and greater than any other straight line drawn through the same point and terminated by the circumferences. The greatest line therefore depends on the distance between the centers of the two circles.

59. Apply Euc. III. 27, i. 6.

60. Let two unequal circles cut one another, and let the line ABC drawn through B, one of the points of intersection, be the line required, such that AB is equal to BC. Join O, O' the centers of the circles, and draw OP, O'P' perpendiculars on ABC, then PB is equal to BP'; through O' draw O'D parallel to PP'; then ODO' is a right-angled triangle, and a semicircle described on OO' as a diameter will pass through the point D. Hence the synthesis. If the line ABC be supposed to move round the point B and its extremities A, C to be in the extremities of the two circles, it is manifest that ABC admits of a maximum.

61. Suppose the thing done, then it will appear that the line joining the points of intersection of the two circles is bisected at right angles by the line joining the centers of the circles. Since the radii are known, the centers of the two circles may be determined.

62. Let the circles intersect in A, B; and let CAD, EBF be any parallels passing through A, B and intercepted by the circles. Join CE, AB, DF. Then the figure CEFD may be proved to be a parallelogram. Whence CAD is equal to EBF.

63. Complete the circle whose segment is ADB; AHB being the other part. Then since the angle ACB is constant, being in a given segment, the sum of the arcs DE and AHB is constant. But AHB is given, hence ED is also given and therefore constant.

64. From A suppose ACD drawn, so that when BD, BC are joined, AD and DB shall together be double of AC and CB together. Then the angles ACD, ADB are supplementary, and hence the angles BCD, BDC are equal, and the triangle BCD is isosceles. Also the angles BCD, BDC are given, hence the triangle BDC is given in species.

Again  $AD + DB = 2.AC + 2.BC$ , or  $CD = AC + BC$ .

Whence, make the triangle  $bdc$  having its angles at  $d, c$  equal to that in the segment BDA; and make  $ca = cd - cb$ , and join  $ab$ . At A make the angle BAD equal to  $bad$ , and AD is the line required.

65. The line drawn from the point of intersection of the two lines to the center of the given circle may be shewn to be constant, and the center of the given circle is a fixed point.

66. This is at once obvious from Euc. III. 36.

67. This follows directly from Euc. III. 36.

68. Each of the lines CE, DF may be proved parallel to the common chord AB.

69. By constructing the figure and joining AC and AD, by Euc. III. 27, it may be proved that the line BC falls on BD.

70. By constructing the figure and applying Euc. I. 8, 4, the truth is manifest.

71. The bisecting line is a common chord to the two circles; join the other extremities of the chord and the diameter in each circle, and the angles in the two segments may be proved to be equal.

72. Apply Euc. III. 27; i. 32, 6.

73. Draw a common tangent at C the point of contact of the circles, and prove AC and CB to be in the same straight line.

74. Let A, B, be the centers, and C the point of contact of the two circles; D, E the points of contact of the circles with the common tangent DE, and CF a tangent common to the two circles at C, meeting DF in E. Join DC, CE. Then DF, FC, FE may be shewn to be equal, and FC to be at right angles to AB.

75. The line must be drawn to the extremities of the diameters which are on opposite sides of the line joining the centers.

76. The sum of the distances of the center of the third circle from the centers of the two given circles, is equal to the sum of the radii of the given circles, which is constant.

77. Let the circles touch at C either externally or internally, and their diameters AC, BC through the point of contact will either coincide or be in the same straight line. CDE any line through C will cut off similar segments from the two circles. For joining AD, BE, the angles in the segments DAC, EBC are proved to be equal.

The remaining segments are also similar, since they contain angles which are supplementary to the angles DAC, EBC.

78. Let the line which joins the centers of the two circles be produced to meet the circumferences, and let the extremities of this line and any other line from the point of contact be joined. From the center of the larger circle draw perpendiculars on the sides of the right-angled triangle inscribed within it.

79. In general, the locus of a point in the circumference of a circle which rolls within the circumference of another, is a curve called the *Hypocycloid*; but to this there is one exception, in which the radius of one of the circles is double that of the other: in this case, the locus is a straight line, as may be easily shewn from the figure.

80. Let A, B be the centers of the circles. Draw AB cutting the circumferences in C, D. On AB take CE, DF each equal to the radius of the required circle: the two circles described with centers A, B, and radii AE, BF, respectively, will cut one another, and the point of intersection will be the center of the required circle.

81. Apply Euc. III. 31.

82. Apply Euc. III. 21.

83. (1) When the tangent is on the same side of the two circles. Join C, C' their centers, and on CC' describe a semicircle. With center C' and radius equal to the *difference* of the radii of the two circles, describe another circle cutting the semicircle in D: join DC' and produce it to meet the circumference of the given circle in B. Through C draw CA parallel to DB and join BA; this line touches the two circles.

(2) When the tangent is on the alternate sides. Having joined C, C'; on CC' describe a semicircle; with center C, and radius equal to the *sum* of the radii of the two circles describe another circle cutting the semicircle in D, join CD cutting the circumference in A, through C draw CB parallel to CA and join AB.

84. The possibility is obvious. The point of bisection of the segment intercepted between the convex circumferences will be the center of one of the circles: and the center of a second circle will be found to be the point of intersection of two circles described from the centers of the given circles with their radii increased by the radius of the second circle. The line passing through the centers of these two circles will be the locus of the centers of all the circles which touch the two given circles.

85. At any points P, R in the circumferences of the circles, whose centers are A, B, draw PQ, RS, tangents equal to the given lines, and join AQ, BS. These being made the sides of a triangle of which AB is the base, the vertex of the triangle is the point required.

86. In each circle draw a chord of the given length, describe circles concentric with the given circles touching these chords, and then draw a straight line touching these circles.

87. Within one of the circles draw a chord cutting off a segment equal to the given segment, and describe a concentric circle touching the chord: then draw a straight line touching this latter circle and the other given circle.

88. The tangent may intersect the line joining the centers, or the line produced. Prove that the angle in the segment of one circle is equal to the angle in the corresponding segment of the other circle.

89. Join the centers  $A, B$ ; at  $C$  the point of contact draw a tangent, and at  $A$  draw  $AF$  cutting the tangent in  $F$ , and making with  $CF$  an angle equal to one-fourth of the given angle. From  $F$  draw tangents to the circles.

90. Let  $C$  be the center of the given circle, and  $D$  the given point in the given line  $AB$ . At  $D$  draw any line  $DE$  at right angles to  $AB$ , then the center of the circle required is in the line  $AE$ . Through  $C$  draw a diameter  $FG$  parallel to  $DE$ , the circle described passing through the points  $E, F, G$  will be the circle required.

91. Apply Euc. III. 18.

92. Let  $A, B$  be the two given points, and  $C$  the center of the given circle. Join  $AC$ , and at  $C$  draw the diameter  $DCE$  perpendicular to  $AC$ , and through the points  $A, D, E$  describe a circle, and produce  $AC$  to meet the circumference in  $F$ . Bisect  $AF$  in  $G$ , and  $AB$  in  $H$ , and draw  $GK, HK$ , perpendiculars to  $AF, AB$  respectively and intersecting in  $K$ . Then  $K$  is the center of the circle which passes through the points  $A, B$ , and bisects the circumference of the circle whose center is  $C$ .

93. Let  $D$  be the given point and  $EF$  the given straight line. (fig. Euc. III. 32.) Draw  $DB$  to make the angle  $DBF$  equal to that contained in the alternate segment. Draw  $BA$  at right angles to  $EF$ , and  $DA$  at right angles to  $DB$  and meeting  $BA$  in  $A$ . Then  $AB$  is the diameter of the circle.

94. Let  $A, B$  be the given points, and  $CD$  the given line. From  $E$  the middle of the line  $AB$ , draw  $EM$  perpendicular to  $AB$ , meeting  $CD$  in  $M$ , and draw  $MA$ . In  $EM$  take any point  $F$ ; draw  $FH$  to make the given angle with  $CD$ ; and draw  $FG$  equal to  $FH$ , and meeting  $MA$  produced in  $G$ . Through  $A$  draw  $AP$  parallel to  $FG$ , and  $CPK$  parallel to  $FH$ . Then  $P$  is the center, and  $C$  the third defining point of the circle required: and  $AP$  may be proved equal to  $CP$  by means of the triangles  $GMP, AMP$ ; and  $HMF, CMP$ , Euc. VI. 2. Also  $CPK$  the diameter makes with  $CD$  the angle  $KCD$  equal to  $FHD$ , that is, to the given angle.

95. Let  $A, B$  be the two given points, join  $AB$  and bisect  $AB$  in  $C$ , and draw  $CD$  perpendicular to  $AB$ , then the center of the required circle will be in  $CD$ . From  $O$  the center of the given circle draw  $CFG$  parallel to  $CD$ , and meeting the circle in  $F$  and  $AB$  produced in  $G$ . At  $F$  draw a chord  $FF'$  equal to the given chord. Then the circle which passes through the points at  $B$  and  $F$ , passes also through  $F'$ .

96. Let the straight line joining the centers of the two circles be produced both ways to meet the circumference of the exterior circle.

97. Let  $A$  be the common center of two circles, and  $BCDE$  the chord such that  $BE$  is double of  $CD$ . From  $A, B$  draw  $AF, BG$  perpendicular to  $BE$ . Join  $AC$ , and produce it to meet  $BG$  in  $G$ . Then  $AC$  may be shewn to be equal to  $CG$ , and the angle  $CBG$  being a right angle, is the angle in the semicircle described on  $CG$  as its diameter.

98. The lines joining the common center and the extremities of the chords of the circles, may be shewn to contain unequal angles, and the angles at the centers of the circles are double the angles at the circumferences, it follows that the segments containing these unequal angles are not similar.

99. Let  $AB, AC$  be the straight lines drawn from  $A$ , a point in



the outer circle to touch the inner circle in the points D, E, and meet the outer circle again at B, C. Join BC, DE. Prove BC double of DE.

Let O be the center, and draw the common diameter AOG intersecting BC in F, and join EF. Then the figure DBFE may be proved to be a parallelogram.

100. This appears from Euc. III. 14.

101. The given point may be either within or without the circle. Draw a chord in the circle equal to the given chord, and describe a concentric circle touching the chord, and through the given point draw a line touching this latter circle.

102. The diameter of the inner circle must not be less than one-third of the diameter of the exterior circle.

103. Suppose AD, DB to be the tangents to the circle AEB containing the given angle. Draw DC to the center C and join CA, CB. Then the triangles ACD, BCD are always equal: DC bisects the given angle at D and the angle ACB. The angles CAB, CBD, being right angles, are constant, and the angles ADC, BDC are constant, as also the angles ACD, BCD; also AC, CB the radii of the given circle. Hence the locus of D is a circle whose center is C and radius CD.

104. Let C be the center of the inner circle; draw any radius CD, at D draw a tangent CE equal to CD, join CE, and with center C and radius CE describe a circle and produce ED to meet the circle again in F.

105. Take C the center of the given circle, and draw any radius CD, at D draw DE perpendicular to DC and equal to the length of the required tangent; with center C and radius CE describe a circle.

106. This is manifest from Euc. III. 36.

107. Let AB, AC be the sides of a triangle ABC. From A draw the perpendicular AD on the opposite side, or opposite side produced. The semicircles described on AB, BC both pass through D. Euc. III. 31.

108. Let A be the right angle of the triangle ABC, the first property follows from the preceding Theorem 107. Let DE, DF be drawn to E, F the centers of the circles on AB, AC and join EF. Then ED may be proved to be perpendicular to the radius DF of the circle on AC at the point D.

109. Let ABC be a triangle, and let the arcs be described on the sides externally containing angles, whose sum is equal to two right angles. It is obvious that the sum of the angles in the remaining segments is equal to four right angles. These arcs may be shewn to intersect each other in one point D. Let  $a, b, c$  be the centers of the circles on BC, AC, AB. Join  $ab, bc, ca$ ;  $Ab, bC, Ca$ ;  $aB, Bc, cA$ ;  $bd, cD, aD$ . Then the angle  $cba$  may be proved equal to one-half of the angle  $AbC$ . Similarly, the other two angles of  $abc$ .

110. It may be remarked, that generally, the mode of proof by which, in pure geometry, three lines must, under specified conditions, pass through the same point, is that by *reductio ad absurdum*. This will for the most part require the converse theorem to be first proved or taken for granted.

The converse theorem in this instance is, "If two perpendiculars drawn from two angles of a triangle upon the opposite sides, intersect in a point, the line drawn from the third angle through this point will be perpendicular to the third side."

The proof will be formally thus: Let EHD be the triangle, AC, BD two perpendiculars intersecting in F. If the third perpendicular EG do not pass through F, let it take some other position as EH; and through F draw EFG to meet AD in G. Then it has been proved that

EG is perpendicular to AD : whence the two angles EHG, EGH of the triangle EGH are equal to two right angles :—which is absurd.

111. The circle described on AB as a diameter will pass through E and D. Then Euc. III. 36.

112. Since all the triangles are on the same base and have equal vertical angles, these angles are in the same segment of a given circle.

The lines bisecting the vertical angles may be shewn to pass through the extremity of that diameter which bisects the base.

113. Let AC be the common base of the triangles, ABC the isosceles triangle, and ADC any other triangle on the same base AC and between the same parallels AC, BD. Describe a circle about ABC, and let it cut AD in E and join EC. Then, Euc. I. 17, III. 21.

114. Let ABC be the given isosceles triangle having the vertical angle at C, and let FG be any given line. Required to find a point P in FG such that the distance PA shall be double of PC. Divide AC in D so that AD is double of DC, produce AC to E and make AE double of AC. On DE describe a circle cutting FG in P, then PA is double of PC. This is found by shewing that  $AP^2 = 4 \cdot PC^2$ .

115. On any two sides of the triangle, describe segments of circles each containing an angle equal to two-thirds of a right angle, the point of intersection of the arcs within the triangle will be the point required, such that three lines drawn from it to the angles of the triangle shall contain equal angles. Euc. III. 22.

116. Let A be the base of the tower, AB its altitude BC the height of the flagstaff, AD a horizontal line drawn from A. If a circle be described passing through the points B, C, and touching the line AD in the point E: E will be the point required. Give the analysis.

117. If the ladder be supposed to be raised in a vertical plane, the locus of the middle point may be shewn to be a quadrantal arc of which the radius is half the length of the ladder.

118. The line drawn perpendicular to the diameter from the other extremity of the tangent is parallel to the tangent drawn at the extremity of the diameter.

119. Apply Euc. III. 21.

120. Let A, B, C, be the centers of the three equal circles, and let them intersect one another in the point D : and let the circles whose centers are A, B intersect each other again in E ; the circles whose centers are B, C in F ; and the circles whose centers are C, A in G. Then FG is perpendicular to DE ; DG to FC ; and DF to GE. Since the circles are equal, and all pass through the same point D, the centers A, B, C are in a circle about D whose radius is the same as the radius of the given circles. Join AB, BC, CA ; then these will be perpendicular to the chords DE, DF, DG. Again, the figures DAGC, DBFC, are equilateral, and hence FG is parallel to AB ; that is, perpendicular to DE. Similarly for the other two cases.

121. Let E be the center of the circle which touches the two equal circles whose centers are A, B. Join AE, BE which pass through the points of contact F, G. Whence AE is equal to EB. Also CD the common chord bisects AB at right angles, and therefore the perpendicular from E on AB coincides with CD.

122. Let three circles touch each other at the point A, and from A let a line ABCD be drawn cutting the circumferences in B, C, D. Let O, O', O'' be the centers of the circles, join BO, CO', DO'', these lines are parallel to one another. Euc. I. 5. 28.

123. Proceed as in Theorem 110, *supra*.

124. The three tangents will be found to be perpendicular to the sides of the triangle formed by joining the centers of the three circles.

125. With center A and any radius less than the radius of either of the equal circles, describe the third circle intersecting them in C and D. Join BC, CD, and prove BC and CD to be in the same straight line.

126. Let ABC be the triangle required; BC the given base, BD the given difference of the sides, and BAC the given vertical angle. Join CD and draw AM perpendicular to CD. Then MAD is half the vertical angle and AMD a right angle: the angle BDC is therefore given, and hence D is a point in the arc of a given segment on BC. Also since BD is given, the point D is given, and therefore the sides BA, AC are given. Hence the synthesis.

127. Let ABC be the required triangle, AD the line bisecting the vertical angle and dividing the base BC into the segments BD, DC. About the triangle ABC describe a circle and produce AD to meet the circumference in E, then the arcs BE, EC are equal.

128. Analysis. Let ABC be the triangle, and let the circle ABC be described about it: draw AF to bisect the vertical angle BAC and meet the circle in F, make AV equal to AC, and draw CV to meet the circle in T; join TB and TF, cutting AB in D; draw the diameter FS cutting BC in R, DR cutting AF in E; join AS, and draw AK, AH perpendicular to FS and BC. Then shew that AD is half the sum, and DB half the difference of the sides AB, AC. Next, that the point F in which AF meets the circumscribing circle is given, also the point E where DE meets AF is given. The points A, K, R, E are in a circle, Euc. III. 22.

Hence,  $KF \cdot FR = AF \cdot FE$ , a given rectangle; and the segment KR, which is equal to the perpendicular AH, being given, RF itself is given. Whence the construction.

129. On AB the given base describe a circle such that the segment AEB shall contain an angle equal to the given vertical angle of the triangle. Draw the diameter EMD cutting AB in M at right angles. At D in ED, make the angle EDC equal to half the given difference of the angles at the base, and let DC meet the circumference of the circle in C. Join CA, CB; ABC is the triangle required. For, make CF equal to CB, and join FB cutting CD in G.

130. Let ABC be the triangle, AD the perpendicular on BC. With center A, and AC the less side as radius, describe a circle cutting the base BC in E, and the longer side AB in G, and BA produced in F, and join AE, EG, FC. Then the angle GFC being half the given angle, BAC is given, and the angle BEG equal to GFC is also given. Likewise BE the difference of the segments of the base, and BG the difference of the sides, are given by the problem. Wherefore the triangle BEG is given (with two solutions). Again, the angle EGB being given, the angle AGE, and hence its equal AEG is given; and hence the vertex A is given, and likewise the line AE equal to AC the shortest side is given. Hence the construction.

131. Let ABC be the triangle, D, E the bisections of the sides AC, AB. Join CE, BD intersecting in F. Bisect BD in G and join EG. Then EF, one-third of EC is given, and BG one-half of BD is also given. Now EG is parallel to AC; and the angle BAC being given, its equal opposite angle BEG is also given. Whence the segment of the circle containing the angle BEG is also given. Hence F is a given point, and FE a given line, whence E is in the circumference of the given circle about F whose radius is FE. Wherefore E being in two given circles, it is itself their given intersection.

132. Of all triangles on the same base and having equal vertical angles, that triangle will be the greatest whose perpendicular from the vertex on the base is a maximum, and the greatest perpendicular is that which bisects the base. Whence the triangle is isosceles.

133. Let  $AB$  be the given base and  $ABC$  the sum of the other two sides; at  $B$  draw  $BD$  at right angles to  $AB$  and equal to the given altitude, produce  $BD$  to  $E$  making  $DE$  equal to  $BD$ . With center  $A$  and radius  $AC$  describe the circle  $CFG$ , draw  $FO$  at right angles to  $BE$  and find in it the center  $O$  of the circle which passes through  $B$  and  $E$  and touches the former circle in the point  $F$ . The centers  $A, O$  being joined and the line produced, will pass through  $F$ . Join  $OB$ . Then  $AOB$  is the triangle required.

134. Since the area and bases of the triangle are given, the altitude is given. Hence the problem is—given the base, the vertical angle and the altitude, describe the triangle.

135. Apply Euc. III. 27.

136. The fixed point may be proved to be the center of the circle.

137. Let the line which bisects any angle  $BAD$  of the quadrilateral, meet the circumference in  $E$ , join  $EC$ , and prove that the angle made by producing  $DC$  is bisected by  $EC$ .

138. Draw the diagonals of the quadrilateral, and by Euc. III. 21, I. 29.

139. From the center draw lines to the angles: then Euc. III. 27.

140. The centers of the four circles are determined by the intersections of the lines which bisect the four angles of the given quadrilateral. Join these four points, and the opposite angles of the quadrilateral so formed are respectively equal to two right angles.

141. Let  $ABCD$  be the required trapezium inscribed in the given circle (fig. Euc. III. 22.) of which  $AB$  is given, also the sum of the remaining three sides and the angle  $ADC$ . Since the angle  $ADC$  is given, the opposite angle  $ABC$  is known, and therefore the point  $C$  and the side  $BC$ . Produce  $AD$  and make  $DE$  equal to  $DC$  and join  $EC$ . Since the sum of  $AD, DC, CB$  is given, and  $DC$  is known, therefore the sum of  $AD, DC$  is given, and likewise  $AC$ , and the angle  $ADC$ . Also the angle  $DEC$  being half of the angle  $ADC$  is given. Whence the segment of the circle which contains  $AEC$  is given, also  $AE$  is given, and hence the point  $E$ , and consequently the point  $D$ . Whence the construction.

142. Let  $ADBC$  be the inscribed quadrilateral; let  $AC, BD$  produced meet in  $O$ , and  $AB, CD$  produced meet in  $P$ , also let the tangents from  $O, P$  meet the circles in  $K, H$  respectively. Join  $OP$ , and about the triangle  $PAC$  describe a circle cutting  $PO$  in  $G$  and join  $AG$ . Then  $A, B, G, O$  may be shewn to be points in the circumference of a circle. Whence the sum of the squares on  $OH$  and  $PK$  may be found by Euc. III. 36, and shewn to be equal to the square on  $OP$ .

143. This will be manifest from the equality of the two tangents drawn to a circle from the same point.

144. Apply Euc. III. 22.

145. A circle can be described about the figure  $AECBF$ .

146. Apply Euc. III. 22, 32.

147. Apply Euc. III. 21, 22, 32.

148. Apply Euc. III. 20, and the angle  $BAD$  will be found to be double of the angles  $CBD$  and  $CDB$  together

149. Let  $ABCD$  be the given quadrilateral figure, and let the angles at  $A, B, C, D$  be bisected by four lines, so that the lines which bisect the angles  $A$  and  $B, B$  and  $C, C$  and  $D, D$  and  $A$ , meet in the points  $a, b, c, d$ , respectively. Prove that the angles at  $a$  and  $c$ , or at  $b$  and  $d$ , are together equal to two right angles.

150. Apply Euc. III. 22.

151. Join the center of the circle with the other extremity of the line perpendicular to the diameter.

152. Let  $AB$  be a chord parallel to the diameter  $FG$  of the circle, fig. Theo. 1, p. 160, and  $H$  any point in the diameter. Let  $HA$  and  $HB$  be joined. Bisect  $FG$  in  $O$ , draw  $OL$  perpendicular to  $FG$  cutting  $AB$  in  $K$ , and join  $HK$ ,  $HL$ ,  $OA$ . Then the square on  $HA$  and  $HF$  may be proved equal to the squares on  $FH$ ,  $HG$  by Theo. 3, p. 114; Euc. I. 47; Euc. II. 9.

153. Let  $A$  be the given point (fig. Euc. III. 36, Cor.) and suppose  $AFC$  meeting the circle in  $F$ ,  $C$ , to be bisected in  $F$ , and let  $AD$  be a tangent drawn from  $A$ . Then  $2 \cdot AF^2 = AF \cdot AC = AD^2$ , but  $AD$  is given, hence also  $AF$  is given. To construct. Draw the tangent  $AD$ . On  $AD$  describe a semicircle  $AGD$ , bisect it in  $G$ ; with center  $A$  and radius  $AG$ , describe a circle cutting the given circle in  $F$ . Join  $AF$  and produce it to meet the circumference again in  $C$ .

154. Let the chords  $AB$ ,  $CD$  intersect each other in  $E$  at right angles. Find  $F$  the center, and draw the diameters  $HEFG$ ,  $AFK$  and join  $AC$ ,  $CK$ ,  $BD$ . Then by Euc. II. 4. 5; III. 35.

155. Let  $E$ ,  $F$  be the points in the diameter  $AB$  equidistant from the center  $O$ ;  $CED$  any chord; draw  $OG$  perpendicular to  $CED$ , and join  $FG$ ,  $OC$ . The sum of the squares on  $DF$  and  $FC$  may be shewn to be equal to twice the square on  $FE$  and the rectangle contained by  $AE$ ,  $EB$  by Euc. I. 47; II. 5; III. 35.

156. Let the chords  $AB$ ,  $AC$  be drawn from the point  $A$ , and let a chord  $FG$  parallel to the tangent at  $A$  be drawn intersecting the chords  $AB$ ,  $AC$  in  $D$  and  $E$ , and join  $BC$ . Then the opposite angles of the quadrilateral  $BDEC$  are equal to two right angles, and a circle would circumscribe the figure. Hence by Euc. I. 36.

157. Let the lines be drawn as directed in the enunciation. Draw the diameter  $AE$  and join  $CE$ ,  $DE$ ,  $BE$ ; then  $AC^2 + AD^2$  and  $2 \cdot AB^2$  may be each shewn to be equal to the square of the diameter.

158. Let  $QOP$  cut the diameter  $AB$  in  $O$ . From  $C$  the center draw  $CH$  perpendicular to  $QP$ . Then  $CH$  is equal to  $OH$ , and by Euc. II. 9, the squares on  $PO$ ,  $OQ$  are readily shewn to be equal to twice the square on  $CP$ .

159. From  $P$  draw  $PQ$  perpendicular on  $AB$  meeting it in  $Q$ . Join  $AC$ ,  $CD$ ,  $DB$ . Then circles would circumscribe the quadrilaterals  $ACPQ$  and  $BDPQ$ , and then by Euc. III. 36.

160. Describe the figure according to the enunciation; draw  $AE$  the diameter of the circle, and let  $P$  be the intersection of the diagonals of the parallelogram. Draw  $EB$ ,  $EP$ ,  $EC$ ,  $EF$ ,  $EG$ ,  $EH$ . Since  $AE$  is a diameter of the circle, the angles at  $F$ ,  $G$ ,  $H$ , are right angles, and  $EF$ ,  $EG$ ,  $EH$  are perpendiculars from the vertex upon the bases of the triangles  $EAB$ ,  $EAC$ ,  $EAP$ . Whence by Euc. II. 13, and theorem 3, page 114, the truth of the property may be shewn.

161. If  $FA$  be the given line (fig. Euc. II. 11), and if  $FA$  be produced to  $C$ ;  $AC$  is the part produced which satisfies the required conditions.

162. Let  $AD$  meet the circle in  $G$ ,  $H$ , and join  $BG$ ,  $GC$ . Then  $BGC$  is a right-angled triangle and  $GD$  is perpendicular to the hypotenuse, and the rectangles may be each shewn to be equal to the square on  $BG$ . Euc. III. 35; II. 5; I. 47. Or, if  $EC$  be joined, the quadrilateral figure  $ADCE$  may be circumscribed by a circle. Euc. III. 31, 22, 36, Cor.

163. On  $PC$  describe a semicircle cutting the given one in  $E$ , and draw  $EF$  perpendicular to  $AD$ ; then  $F$  is the point required.

164. Let AB be the given straight line. Bisect AB in C and on AB as a diameter describe a circle; and at any point D in the circumference, draw a tangent DE equal to a side of the given square; join DC, EC, and with center C and radius CE describe a circle cutting AB produced in F. From F draw FG to touch the circle whose center is C in the point G.

165. Let AD, DF be two lines at right angles to each other, O the centre of the circle BFQ; A any point in AD from which tangents AB, AC are drawn; then the chord BC shall always cut FD in the same point P, wherever the point A is taken in AD. Join AP; then BAC is an isosceles triangle,

$$\text{nd } FD \cdot DE + AD^2 = AB^2 = BP \cdot PC + AP^2 = BP \cdot PC + AD^2 + DP^2, \\ \text{wherefore } BP \cdot PC = FD \cdot DE - DP^2.$$

The point P, therefore, is independent of the position of the point A; and is consequently the same for all positions of A in the line AD.

166. The point E will be found to be that point in BC, from which two tangents to the circles described on AB and CD as diameters, are equal, Euc. III. 36.

167. If AQ, A'P' be produced to meet, these lines with AA' form a right-angled triangle, then Euc. I. 47.

## GEOMETRICAL EXERCISES ON BOOK IV.

### HINTS, &c.

5. Let AB be the given line. Draw through C the center of the given circle the diameter DCE. Bisect AB in F and join FC. Through A, B draw AG, BH parallel to FC and meeting the diameter in G, H: at G, H draw GK, HL perpendicular to DE and meeting the circumference in the points K, L; join KL; then KL is equal and parallel to AB.

6. Trisect the circumference and join the center with the points of trisection.

7. See Euc. IV. 4, 5.

8. Let a line be drawn from the third angle to the point of intersection of the two lines; and the three distances of this point from the angles may be shewn to be equal.

9. Let the line AD drawn from the vertex A of the equilateral triangle, cut the base BC, and meet the circumference of the circle in D. Let DB, DC be joined: AD is equal to DB and DC. If on DA, DE be taken equal to DB, and BE be joined; BDE may be proved to be an equilateral triangle, also the triangle ABE may be proved equal to the triangle CBD.

The other case is when the line does not cut the base.

10. Let a circle be described upon the base of the equilateral triangle, and let an equilateral triangle be inscribed in the circle. Draw a diameter from one of the vertices of the inscribed triangle, and join the other extremity of the diameter with one of the other extremities of the sides of the inscribed triangle. The side of the inscribed triangle may then be proved to be equal to the perpendicular in the other triangle.

11. The line joining the points of bisection, is parallel to the base of the triangle and therefore cuts off an equilateral triangle from the given triangle. By Euc. III. 21; I. 6, the truth of the theorem may be shewn.

12. Let a diameter be drawn from any angle of an equilateral tri-

angle inscribed in a circle to meet the circumference. It may be proved that the radius is bisected by the opposite side of the triangle.

13. Let  $ABC$  be an equilateral triangle inscribed in a circle, and let  $AB'C'$  be an isosceles triangle inscribed in the same circle, having the same vertex  $A$ . Draw the diameter  $AD$  intersecting  $BC$  in  $E$ , and  $B'C'$  in  $E'$ , and let  $B'C'$  fall below  $BC$ . Then  $AB$ ,  $BE$ , and  $AB'$ ,  $B'E'$ , are respectively the semi-perimeters of the triangles. Draw  $BF$  perpendicular to  $BC$ , and cut off  $AH$  equal to  $AB$ , and join  $BH$ . If  $BF$  can be proved to be greater than  $B'H$ , the perimeter of  $ABC$  is greater than the perimeter of  $AB'C'$ . Next let  $B'C'$  fall above  $BC$ .

14. The angles contained in the two segments of the circle, may be shewn to be equal, then by joining the extremities of the arcs, the two remaining sides may be shewn to be parallel.

15. It may be shewn that four equal and equilateral triangles will form an equilateral triangle of the same perimeter as the hexagon, which is formed by six equal and equilateral triangles.

16. Let the figure be constructed. By drawing the diagonals of the hexagon, the proof is obvious.

17. By Euc. I. 47, the perpendicular distance from the center of the circle upon the side of the inscribed hexagon may be found.

18. The alternate sides of the hexagon will fall upon the sides of the triangle, and each side will be found to be equal to one-third of the side of the equilateral triangle.

19. A regular duodecagon may be inscribed in a circle by means of the equilateral triangle and square, or by means of the hexagon. The area of the duodecagon is three times the square on the radius of the circle, which is the square on the side of an equilateral triangle inscribed in the same circle. Theorem 1, p. 196.

20. In general, three straight lines when produced will meet and form a triangle, except when all three are parallel or two parallel are intersected by the third. This Problem includes Euc. IV. 5, and all the cases which arise from producing the sides of the triangle. The circles described touching a side of a triangle and the other two sides produced, are called the *escribed* circles.

21. This is manifest from Euc. III. 21.

22. The point required is the center of the circle which circumscribes the triangle. See the notes on Euc. III. 20, p. 155.

23. If the perpendiculars meet the three sides of the triangle, the point is within the triangle, Euc. IV. 4. If the perpendiculars meet the base and the two sides produced, the point is the center of the *escribed* circle.

24. This is manifest from Euc. III. 11, 18.

25. The base  $BC$  is intersected by the perpendicular  $AD$ , and the side  $AC$  is intersected by the perpendicular  $BE$ . From Theorem I. p. 160; the arc  $AF$  is proved equal to  $AE$ , or the arc  $FE$  is bisected in  $A$ . In the same manner the arcs  $FD$ ,  $DE$ , may be shewn to be bisected in  $B$ .

26. Let  $ABC$  be a triangle, and let  $D$ ,  $E$  be the points where the inscribed circle touches the sides  $AB$ ,  $AC$ . Draw  $BE$ ,  $CD$  intersecting each other in  $O$ . Join  $AO$ , and produce it to meet  $BC$  in  $F$ . Then  $F$  is the point where the inscribed circle touches the third side  $BC$ . If  $F$  be not the point of contact, let some other point  $G$  be the point of contact. Through  $D$  draw  $DH$  parallel to  $AC$ , and  $DK$  parallel to  $BC$ . By the similar triangles,  $CG$  may be proved equal to  $CF$ , or  $G$  the point of contact coincides with  $F$ , the point where the line drawn from  $A$  through  $O$  meets  $BC$ .

27. In the figure, *Euc. iv. 5*. Let *AF* bisect the angle at *A*, and be produced to meet the circumference in *G*. Join *GB*, *GC* and find the center *H* of the circle inscribed in the triangle *ABC*. The lines *GH*, *GB*, *GC* are equal to one another.

28. Let *ABC* be any triangle inscribed in a circle, and let the perpendiculars *AD*, *BE*, *CF* intersect in *G*. Produce *AD* to meet the circumference in *H*, and join *BH*, *CH*. Then the triangle *BHC* may be shewn to be equal in all respects to the triangle *BGC*, and the circle which circumscribes one of the triangles will also circumscribe the other. Similarly may be shewn by producing *BE* and *CF*, &c.

29. First. Prove that the perpendiculars *Aa*, *Bb*, *Cc* pass through the same point *O*, as *Theo. 112*, p. 171. Secondly. That the triangles *Acb*, *Bca*, *Cab* are equiangular to *ABC*. *Euc. iii. 21*. Thirdly. That the angles of the triangle *abc* are bisected by the perpendiculars; and lastly, by means of *Prob. 4*, p. 71, that  $ab + bc + ca$  is a minimum.

30. The equilateral triangle can be proved to be the least triangle which can be circumscribed about a circle.

31. Through *C* draw *CH* parallel to *AB* and join *AH*. Then *HAC* the difference of the angles at the base is equal to the angle *HFC*. *Euc. iii. 21*, and *HFC* is bisected by *FG*.

32. Let *F*, *G*, (*figure, Euc. iv. 5*), be the centers of the circumscribed and inscribed circles; join *GF*, *GA*, then the angle *GAF* which is equal to the difference of the angles *GAD*, *FAD*, may be shewn to be equal to half the difference of the angles *ABC* and *ACB*.

33. This Theorem may be stated more generally, as follows:

Let *AB* be the base of a triangle, *AEB* the locus of the vertex; *D* the bisection of the remaining arc *ADB* of the circumscribing circle; then the locus of the center of the inscribed circle is another circle whose center is *D* and radius *DB*. For join *CD*: then *P* the center of the inscribed circle is in *CD*. Join *AP*, *PB*; then these lines bisect the angles *CAB*, *CBA*, and *DB*, *DP*, *DA* may be proved to be equal to one another.

34. Let *ABC* be a triangle, having *C* a right angle, and upon *AC*, *BC*, let semicircles be described: bisect the hypotenuse in *D*, and let fall *DE*, *DF* perpendiculars on *AC*, *BC* respectively, and produce them to meet the circumferences of the semicircles in *P*, *Q*; then *DP* may be proved to be equal to *DQ*.

35. Let the angle *BAC* be a right angle, *fig. Euc. iv. 4*. Join *AD*. Then *Euc. iii. 17*, note p. 155.

36. Suppose the triangle constructed, then it may be shewn that the difference between the hypotenuse and the sum of the two sides is equal to the diameter of the inscribed circle.

37. Let *P*, *Q* be the middle points of the arcs *AB*, *AC*, and let *PQ* be joined, cutting *AB*, *AC* in *DE*; then *AD* is equal to *AE*. Find the center *O* and join *OP*, *QO*.

38. With the given radius of the circumscribed circle, describe a circle. Draw *BC* cutting off the segment *BAC* containing an angle equal to the given vertical angle. Bisect *BC* in *D*, and draw the diameter *EDF*: join *FB*, and with center *F* and radius *FB* describe a circle: this will be the locus of the centers of the inscribed circle (see *Theorem 33, supra*.) On *DE* take *DG* equal to the given radius of the inscribed circle, and through *G* draw *GH* parallel to *BC*, and meeting the locus of the centers in *H*. *H* is the center of the inscribed circle.

39. This may readily be effected in almost a similar way to the preceding Problem.

40. With the given radius describe a circle, then by *Euc. iii. 34*.



41. Let  $ABC$  be a triangle on the given base  $BC$  and having its vertical angle  $A$  equal to the given angle. Then since the angle at  $A$  is constant,  $A$  is a point in the arc of a segment of a circle described on  $BC$ . Let  $D$  be the center of the circle inscribed in the triangle  $ABC$ . Join  $DA, DB, DC$ : then the angles at  $B, C, A$ , are bisected. Euc. iv. 4. Also since the angles of each of the triangles  $ABC, DBC$  are equal to two right angles, it follows that the angle  $BDC$  is equal to the angle  $A$  and half the sum of the angles  $B$  and  $C$ . But the sum of the angles  $B$  and  $C$  can be found, because  $A$  is given. Hence the angle  $BDC$  is known, and therefore  $D$  is the locus of the vertex of a triangle described on the base  $BC$  and having its vertical angle at  $D$  double of the angle at  $A$ .

42. Suppose the parallelogram to be rectangular and inscribed in the given triangle and to be equal in area to half the triangle: it may be shewn that the parallelogram is equal to half the altitude of the triangle, and that there is a restriction to the magnitude of the angle which two adjacent sides of the parallelogram make with one another.

43. Let  $ABC$  be the given triangle, and  $A'B'C'$  the other triangle, to the sides of which the inscribed triangle is required to be parallel. Through any point  $a$  in  $AB$  draw  $ab$  parallel to  $A'B'$  one side of the given triangle and through  $a, b$  draw  $ac, bc$  respectively parallel to  $AC, BC$ . Join  $Ac$  and produce it to meet  $BC$  in  $D$ ; through  $D$  draw  $DE, DF$ , parallel to  $ca, cb$ , respectively, and join  $EF$ . Then  $DEF$  is the triangle required.

44. This point will be found to be the intersection of the diagonals of the given parallelogram.

45. The difference of the two squares is obviously the sum of the four triangles at the corners of the exterior square.

46. (1) Let  $ABCD$  be the given square: join  $AC$ , at  $A$  in  $AC$ , make the angles  $CAE, CAF$ , each equal to one-third of a right angle, and join  $EF$ .

(2) Bisect  $AB$  any side in  $P$ , and draw  $PQ$  parallel to  $AD$  or  $BC$ , then at  $P$  make the angles as in the former case.

47. Each of the interior angles of a regular octagon may be shewn to be equal to three-fourths of two right angles, and the exterior angles made by producing the sides, are each equal to one-fourth of two right angles, or one-half of a right angle.

48. Let the diagonals of the rhombus be drawn; the center of the inscribed circle may be shewn to be the point of their intersection.

49. Let  $ABCD$  be the required square. Join  $O, O'$  the centers of the circles and draw the diagonal  $AEC$  cutting  $OO'$  in  $E$ . Then  $E$  is the middle point of  $OO'$  and the angle  $AEO$  is half a right angle.

50. Let the squares be inscribed in, and circumscribed about a circle, and let the diameters be drawn, the relation of the two squares is manifest.

51. Let one of the diagonals of the square be drawn, then the isosceles right-angled triangle which is half the square, may be proved to be greater than any other right-angled triangle upon the same hypotenuse.

52. Take half of the side of the square inscribed in the given circle, this will be equal to a side of the required octagon. At the extremities on the same side of this line make two angles each equal to three-fourths of two right angles, bisect these angles by two straight lines, the point at which they meet will be the center of the circle which circumscribes the octagon, and either of the bisecting lines is the radius of the circle.

53. First shew the possibility of a circle circumscribing such a figure, and then determine the center of the circle.

54. By constructing the figures and drawing lines from the center of

the circle to the angles of the octagon, the areas of the eight triangles may be easily shewn to be equal to eight times the rectangle contained by the radius of the circle, and half the side of the inscribed square.

55. Let  $AB, AC, AD$ , be the sides of a square, a regular hexagon and an octagon respectively inscribed in the circle whose center is  $O$ . Produce  $AC$  to  $E$  making  $AE$  equal to  $AB$ ; from  $E$  draw  $EF$  touching the circle in  $F$ , and prove  $EF$  to be equal to  $AD$ .

56. Let the circle required touch the given circle in  $P$ , and the given line in  $Q$ . Let  $C$  be the center of the given circle and  $C'$  that of the required circle. Join  $CC', C'Q, QP$ ; and let  $QP$  produced meet the given circle in  $R$ , join  $RC$  and produce it to meet the given line in  $V$ . Then  $RCV$  is perpendicular to  $VQ$ . Hence the construction.

57. Let  $A, B$  be the centers of the given circles and  $CD$  the given straight line. On the side of  $CD$  opposite to that on which the circles are situated, draw a line  $EF$  parallel to  $CD$  at a distance equal to the radius of the smaller circle. From  $A$  the center of the larger circle describe a concentric circle  $GH$  with radius equal to the difference of the radii of the two circles. Then the center of the circle touching the circle  $GH$ , the line  $EF$ , and passing through the center of the smaller circle  $B$ , may be shewn to be the center of the circle which touches the circles whose centers are  $A, B$ , and the line  $CD$ .

58. Let  $AB, CD$  be the two lines given in position and  $E$  the center of the given circle. Draw two lines  $FG, HI$  parallel to  $AB, CD$  respectively and external to them. Describe a circle passing through  $E$  and touching  $FG, HI$ . Join the centers  $E, O$ , and with center  $O$  and radius equal to the difference of the radii of these circles describe a circle; this will be the circle required.

59. Let the circle  $ACF$  having the center  $G$ , be the required circle touching the given circle whose center is  $B$ , in the point  $A$ , and cutting the other given circle in the point  $C$ . Join  $BG$ , and through  $A$  draw a line perpendicular to  $BG$ ; then this line is a common tangent to the circles whose centers are  $B, G$ . Join  $AC, GC$ . Hence the construction.

60. Let  $C$  be the given point in the given straight line  $AB$ , and  $D$  the center of the given circle. Through  $C$  draw a line  $CE$  perpendicular to  $AB$ ; on the other side of  $AB$ , take  $CE$  equal to the radius of the given circle. Draw  $ED$ , and at  $D$  make the angle  $EDF$  equal to the angle  $DEC$ , and produce  $EC$  to meet  $DF$ . This gives the construction for one case, when the given line does not cut or touch the other circle.

61. This is a particular case of the general problem; To describe a circle passing through a given point and touching two straight lines given in position.

Let  $A$  be the given point between the two given lines which when produced meet in the point  $B$ . Bisect the angle at  $B$  by  $BD$  and through  $A$  draw  $AD$  perpendicular to  $BD$  and produce it to meet the two given lines in  $C, E$ . Take  $DF$  equal to  $DA$ . and on  $CB$  take  $CG$  such that the rectangle contained by  $CF, CA$  is equal to the square on  $CG$ . The circle described through the points  $F, A, G$ , will be the circle required. Deduce the particular case when the given lines are at right angles to one another, and the given point in the line which bisects the angle at  $B$ . If the lines are parallel, when is the solution possible?

62. Let  $A, B$ , be the centers of the given circles, which touch externally in  $E$ ; and let  $C$  be the given point in that whose center is  $B$ . Make  $CD$  equal to  $AE$  and draw  $AD$ ; make the angle  $DAG$  equal to the angle  $ADG$ : then  $G$  is the center of the circle required, and  $GC$  its radius.

63. If the three points be such as when joined by straight lines a triangle is formed; the points at which the inscribed circle touches the sides of the triangle, are the points at which the three circles touch one another. Euc. iv. 4. Different cases arise from the relative position of the three points.

64. Bisect the angle contained by the two lines at the point where the bisecting line meets the circumference, draw a tangent to the circle and produce the two straight lines to meet it. In this triangle inscribe a circle.

65. From the given angle draw a line through the center of the circle, and at the point where the line intersects the circumference, draw a tangent to the circle, meeting two sides of the triangle. The circle inscribed within this triangle will be the circle required.

66. Let the diagonal AD cut the arc in P, and let O be the center of the inscribed circle. Draw OQ perpendicular to AB. Draw PE a tangent at P meeting AB produced in E: then BE is equal to PD. Join PQ, PB. Then AB may be proved equal to QE. Hence AQ is equal to BE or DP.

67. Suppose the center of the required circle to be found, let fall two perpendiculars from this point upon the radii of the quadrant, and join the center of the circle with the center of the quadrant and produce the line to meet the arc of the quadrant. If three tangents be drawn at the three points thus determined in the two semicircles and the arc of the quadrant, they form a right-angled triangle which circumscribes the required circle.

68. Let AB be the base of the given segment, C its middle point. Let DCE be the required triangle having the sum of the base DE and perpendicular CF equal to the given line. Produce CF to H making FH equal to DE. Join HD and produce it, if necessary, to meet AB produced in K. Then CK is double of DF. Draw DL perpendicular to CK.

69. From the vertex of the isosceles triangle let fall a perpendicular on the base. Then, in each of the triangles so formed, inscribe a circle, Euc. iv. 4; next inscribe a circle so as to touch the two circles and the two equal sides of the triangle. This gives one solution: the problem is indeterminate.

70. If BD be shewn to subtend an arc of the larger circle equal to one-tenth of the whole circumference:—then BD is a side of the decagon in the larger circle. And if the triangle ABD can be shewn to be inscribable in the smaller circle, BD will be the side of the inscribed pentagon.

71. It may be shewn that the angles ABF, BFD stand on two arcs, one of which is three times as large as the other.

72. It may be proved that the diagonals bisect the angles of the pentagon, and the five-sided figure formed by their intersection, may be shewn to be both equiangular and equilateral.

73. The figure ABCDE is an irregular pentagon inscribed in a circle; it may be shewn that the five angles at the circumference stand upon arcs whose sum is equal to the whole circumference of the circle; Euc. iii. 20.

74. If a side CD (figure, Euc. iv. 11) of a regular pentagon be produced to K, the exterior angle ADK of the inscribed quadrilateral figure ABCD is equal to the angle ABC, one of the interior angles of the pentagon. From this a construction may be made for the method of folding the ribbon.

75. In the figure, Euc. iv. 10, let DC be produced to meet the circumference in F, and join FB. Then FB is the side of a regular pentagon inscribed in the larger circle, D is the middle of the arc subtended by the adjacent side of the pentagon. Then the difference of FD and BD is equal to the radius AB. Next, it may be shewn, that FD is divided in the same manner in C as AB, and by Euc. ii. 4, 11, the squares on FD and DB are three times the square on AB, and the rectangle of FD and DB is equal to the square on AB.

76. If one of the diagonals be drawn, this line with three sides of the pentagon forms a quadrilateral figure of which three consecutive sides are equal. The problem is reduced to the inscription of a quadrilateral in a square.

77. This may be deduced from Euc. iv. 11.

78. The angle at A the center of the circle (fig. Euc. iv. 10.) is one-tenth of four right angles, the arc BD is therefore one-tenth of the circumference, and the chord BD is the side of a regular decagon inscribed in the larger circle. Produce DC to meet the circumference in F and join BF, then BF is the side of the inscribed pentagon, and AB is the side of the inscribed hexagon. Join FA. Then FCA may be proved to be an isosceles triangle and FB is a line drawn from the vertex meeting the base produced. If a perpendicular be drawn from F on BC, the difference of the squares on FB, FC may be shewn to be equal to the rectangle AB, BC, (Euc. i. 47; ii. 5. Cor.); or the square on AC.

79. Divide the circle into three equal sectors, and draw tangents to the middle points of the arcs, the problem is then reduced to the inscription of a circle in a triangle.

80. Let the inscribed circles whose centers are A, B touch each other in G, and the circle whose center is C, in the points D, E; join A, D; A, E; at D, draw DF perpendicular to DA, and EF to EB, meeting in F. Let F, G be joined, and FG be proved to touch the two circles in G whose centers are A and B.

81. The problem is the same as to find how many equal circles may be placed round a circle of the same radius, touching this circle and each other. The number is six.

82. This is obvious from Euc. iv. 7, the side of a square circumscribing a circle being equal to the diameter of the circle.

83. Each of the vertical angles of the triangles so formed, may be proved to be equal to the difference between the exterior and interior angle of the heptagon.

84. Every regular polygon can be divided into equal isosceles triangles by drawing lines from the center of the inscribed or circumscribed circle to the angular points of the figure, and the number of triangles will be equal to the number of sides of the polygon. If a perpendicular FG be let fall from F (figure, Euc. iv. 14) the center on the base CD of FCD, one of these triangles, and if GF be produced to H till FH be equal to FG, and HC, HD be joined, an isosceles triangle is formed, such that the angle at H is half the angle at F. Bisect HC, HD in K, L, and join KL; then the triangle HKL may be placed round the vertex H, twice as many times as the triangle CFD round the vertex F.

85. The sum of the arcs on which stand the 1st, 3rd, 5th, &c. angles, is equal to the sum of the arcs on which stand the 2nd, 4th, 6th, &c. angles.

86. The proof of this property depends on the fact, that an isosceles triangle has a greater area than any scalene triangle of the same perimeter.

# INDEX

TO THE

## PROBLEMS AND THEOREMS

IN THE

### GEOMETRICAL EXERCISES.

#### ABBREVIATIONS.

Senate House Examination for Degrees.	St. Catharine's College. Cath.
S. H.	Jesus College. Jes.
Smith's Mathematical Prizes. S. P.	Christ's College. Chr.
Bell's University Scholarships. B. S.	St. John's College. Joh.
St Peter's College. Pet.	Magdalene College. Mag.
Clare College. Cla.	Trinity College. Trin.
Pembroke College. Pem.	Emmanuel College. Emm.
Gonville and Caius College. Cai.	Sidney Sussex College. Sid.
Trinity Hall. T. H.	Downing College. Down.
Corpus Christi College. C. C.	In the years the centuries are omitted,
King's College. Ki.	and the places are supplied by a comma
Queen's College. Qu.	prefixed, thus, 45 means 1845.

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1 Emm., 22, 35, 46.	8 Qu., 26, 28. S. H.	27 Chr., 26, 41, 52
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2 Trin., 40. Cai., 57.	Emm., 50. C. C.	Pet., 38. Trin., 39,
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3 Trin., 32, 37, 50.	9 Mag., 38. Joh., 58.	28 S. H., 58.
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4 Sid., 30, 43. Jes.	12 S. H., 40, 54.	30 C. C., 53. Qu., 54.
, 50, 58. Qu., 34.	13 Cath., 31. S. H., 50.	Chr., 56.
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Emm., 56.	15 Pet., 57.	32 S. H., 36, 48. Mag.
5 Emm., 21. Qu., 23.	16 Cath., 22, 33. Trin.	47. Chr., 54.
, 40, 42. Trin., 26.	, 37.	33 Emm., 25.
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, 55. Pem., 32, 38.	18 Chr., 56.	35 Chr., 28. Pem., 42.
6 S. H., 17. Trin.	19 Emm., 56.	Jes., 51.
, 24, 37. Qu., 25.	20 T. H., 51.	36 Trin., 26. Sid., 43.
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7 S. H., 19. Trin.	, 57.	39 Qu., 31. Cath., 35.
, 29. Qu., 35. Pem.	24 Jes., 58.	Emm., 35. Sid.
, 44. Jes., 49. B. S.	25 S. H., 14. Cla., 55.	, 38. B. S., 40.
, 55.	26 Cai., 41.	Trin., 27.

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| 40 Trin., 34.            | 86 Cla., 57.             | 121 Pet., 38. Chr., 39.   |
| 41 S. H., 55.            | 87 Pet., 46.             | 122 S. H., 53.            |
| 42 S. H., 04. C. C.      | 88 C. C., 50. Cai., 53.  | 123 Joh., 58.             |
| 23. Chr., 29., 50.       | 89 Mag., 51., 58.        | 124 C. C., 46.            |
| Cath., 35. Jes., 52.     | 90 Jes., 54.             | 125 Pem., 46.             |
| Pet., 36. Qu., 39.       | 91 Cath., 49. S. H., 54. | 126 S. H., 54.            |
| Trin., 37., 49. Cai.     | 92 Jes., 55.             | 127 Emm., 31. Chr.        |
| 40. Pem., 48.            | 93 Cai., 46.             | 38.                       |
| 43 Trin., 54. Emm., 54.  | 94 Jes., 41.             | 128 Trin., 48. Cath., 46. |
| 44 Trin., 58.            | 95 Chr., 43.             | 129                       |
| 45 Cai., 55.             | 96 Joh., 31.             | 130 Pem., 47.             |
| 46 Pet., 58.             | 97 Cai., 36. Cath., 55.  | 131 Cla.                  |
| 47 Chr., 55.             | 98 Emm., 30. Cath.       | 132 Ki., 48. S. H., 53.   |
| 48 Cai., 49.             | 57.                      | Chr., 55., 57.            |
| 49 Jes., 54.             | 99 Trin., 59.            | 133 Qu., 30. Chr., 46.    |
| 50 S. H., 53.            | 100 Pet., 51.            | 134 Pet., 58.             |
| 51 Trin., 39., 51. Pem.  | 101 Qu., 29., 35., 37.   | 135 Jes., 57.             |
| 51.                      | B. S., 39.               | 136 Mag., 57.             |
| 52 Trin., 43.            | 102                      | 137 Cai., 52.             |
| 53 Joh., 26. Pem., 47.   | 103 Mag., 52.            | 138 Trin., 37., 50., 52.  |
| Chr., 52., 53.           | 104 Chr., 47. Cla., 48.  | Joh., 47. Emm.            |
| 54 Cai., 46. Qu., 48.    | 105 Pet., 51.            | 52., 53., 56. Chr.        |
| 55 Cai., 31. Joh., 30.   | 106 Sid., 45. Chr., 47.  | 50. T. H., 52.            |
| 56                       | Emm., 47.                | 139 Cla., 36.             |
| 57 Jes., 52. Cai., 56.   | 107 S. H., 52.           | 140 Mag., 49.             |
| 58 Jes., 55.             | 108 Emm., 57.            | 141 Cla., 36.             |
| 59 Pet., 51.             | 109 S. H., 04. Cai.      | 142 Joh., 58. Chr., 58.   |
| 60 Chr., 39.             | 34. Emm., 39.            | 143 Trin., 53., 54.       |
| 61 Pet., 36.             | 110 Qu., 25. Trin.       | 144 Joh., 16. Qu., 30.    |
| 62 Trin., 52., 54. T. H. | 25., 38. Pet., 39.       | Pem., 33., 49. Jes.       |
| 52.                      | Jes., 52. Pem., 42.      | 46. Trin., 47., 58.       |
| 63 Pet., 51.             | 111 S. H., 03., 18.      | C. C., 58.                |
| 64 Trin., 51.            | Trin., 25., 44. Cla.     | 145 Pet., 27.             |
| 65 Jes., 54.             | 31., 36.                 | 146 S. H., 33.            |
| 66 Pet., 51.             | 112 Qu., 29., 37., 26.   | 147 Chr., 54.             |
| 67 S. H., 48.            | Trin., 27., 33., 36.     | 148 Cla., 56.             |
| 68                       | 40., 49., 50. Chr.       | 149 Jes., 20. Qu., 31.    |
| 69 T. H., 54.            | 44. Pem., 46.            | 48. Cath., 35.            |
| 70 Trin., 40.            | Cath., 58. Emm.          | S. H., 59.                |
| 71                       | 54. Jes., 52. S. H.      | 150 Trin., 40.            |
| 72 Cai., 33. Qu., 33.    | 50. C. C., 58.           | 151 Pet., 32., 35.        |
| 73 Trin., 49.            | 113 Emm., 32.            | 152 Pet., 49.             |
| 74 Qu., 31. Chr., 56.    | 114 Qu., 19., 37. Emm.   | 153 S. H., 55.            |
| Sid., 36. Pet., 53.      | 25., 53. Mag., 29.       | 154 Jes., 53.             |
| 75 Qu., 19.              | 32. Cai., 34. Trin.      | 155 Chr., 56.             |
| 76 Qu., 24.              | 37., 38. Pet., 44.       | 156 T. H., 52.            |
| 77 Cla., 51.             | 52.                      | 157 Joh., 20. Emm.        |
| 78 Qu., 32. Jes., 36.    | 115 S. H., 48.           | 26.                       |
| S. H., 49., 50.          | 116 Qu., 39. Mag., 54.   | 158 Sid., 46. Mag., 58.   |
| 79 S. H., 49. Mag., 52.  | S. H., 59.               | 159 Cai., 37.             |
| 80 Qu., 37.              | 117 Trin., 29.           | 160 Emm., 32. Qu.         |
| 81 Trin., 48.            | 118 Emm., 22. C. C.      | 35., 59. C. C.            |
| 82 Chr., 58.             | 58.                      | 36., 59. Mag., 39.        |
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| 84 Trin., 52.            | 120 S. H., 35., 48. Joh. | 161 Trin., 21., 50.       |
| 85 Cath., 49.            | 37.                      | 162 Jes., 36.             |

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| 1 S. H., 14., 50. Joh.  | 20 Emm., 56.             | 45 Emm., 23., 26., 28. |
| 18. Trin., 35. Chr.     | 21                       | 43., 51. Trin., 27.    |
| 55.                     | 22 Qu., 50.              | 44., 49., 50. Pet.     |
| 2 Joh., 17.             | 23 Qu., 24.              | 30. Mag., 33., 43.     |
| 3 S. H., 16., 59. Trin. | 24 Chr., 49.             | 46., 52. B. S., 38.    |
| 27., 30., 37., 47.      | 25 S. H., 10., 04. Trin. | C. C., 51. Chr.        |
| 48. Mag., 31., 43.      | 29.                      | 41., 47., 50.          |
| Pet., 29., 38. Sid.     | 26 Pet., 43.             | 46 Emm., 28. Sid.      |
| 34. Emm., 21., 27.      | 27 Chr., 49.             | 33. C. C., 39.         |
| 37., 44. Cai., 43.      | 28 Qu., 55.              | 47 Joh., 19. Qu., 29.  |
| Qu., 37. T. H., 55.     | 29 Qu., 57.              | 30., 48.               |
| 4 Emm., 34. Pem.        | 30 Qu., 51.              | 48 Chr., 30. Emm.      |
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| 5 S. H., 03. Joh., 18.  | 32 Cai., 59.             | Cath., 52.             |
| Qu., 21. Trin., 37.     | 33 Chr., 57.             | 49 S. H., 07. T. H.    |
| Sid., 42. Chr., 45.     | 34 Cai., 44.             | 44. Pem., 52. Joh.     |
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| 6 Pet., 52.             | 36 Trin., 49. Cai., 57.  | Emm., 52. S. H.        |
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| 10 Cai., 53.            | 52. S. H., 53.           | 51 Chr., 51.           |
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| 17 Qu., 37.             | 43 Joh., 26. Jes., 37.   | 58 S. H., 59.          |
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| 48. Sid., 49. Pet.      |                          |                        |
| 55. Pem., 58.           |                          |                        |

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| Mag., 53.              | 12 S. H., 43.           | 23 Joh., 17.        |
| 3 Trin., 27.           | 13 S. H., 48.           | 24 Joh., 21.        |
| 4 Mag., 53.            | 14 Cath., 53.           | 25 Chr., 27.        |
| 5 S. H., 04. Sid., 41. | 15                      | 26 S. H., 48.       |
| 6 Trin., 19., 23. Qu.  | 16                      | 27 Pet., 47.        |
| 21., 22. Pem., 30.     | 17 Joh., 57.            | 28 S. H., 49.       |
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| 31. Emm., 34., 42.     | Cai., 34. Emm., 34.     | 30 Emm., 54.        |
| 44. T. H., 54.         | Qu., 36. S. H., 53.     | 31 S. H., 53.       |
| 7 Emm., 22. Pem.       | Chr., 56. Joh., 57.     | 32 S. H., 59.       |
| 36. Joh., 57. S. H.    | 58.                     | 33 Qu., 57.         |
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| 8 Pet., 29. Cla., 46.  | 20 C. C., 42.           |                     |

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- 40 Qu., 36.
- 41 Joh., 28. Qu., 35.
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- 45 Qu., 58.
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- 50 Trin., 34.
- 51 Cai., 32., 41.
- 52 Emm., 21. Pem.  
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- 54 Cla., 56.
- 55 S. H., 55.
- 56 Cla., 56.
- 57 Emm., 57.
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- 68 Cai., 43. Emm., 44.
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| 4 Joh., 16. Pet., 36.    | 36.                     | 68 C. C., 44.            |
| 5 Trin., 31.             | 36 Jes., 31.            | 69 Qu., 44.              |
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| 7 Trin., 37. Jes., 47.   | 38 Joh., 18.            | 37.                      |
| 8                        | 39 Joh., 17. Trin., 36. | 71 Cai., 40.             |
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| 10 Chr., 27.             | 43 Joh., 25.            | 75 C. C., 24.            |
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